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A COURSE IN

INTERPOLATION AND NUMERICAL INTEGRATION

FOR THE MATHEMATICAL LABORATORY

by

DAVID GIBB, M.A., B.Sc.

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AND

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Lecturer in Mathematics in the University of Edinburgh

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PREFACE

THE present work is intended for the use of students who are learning the practice of Interpolation and Numerical Integration. The advantages of a practical knowedge of this part of Mathematics are so obvious that it is needless to insist on them here: and these subjects form an important part of the course in the modern Mathematical Laboratory. There are, however, so many claims on the time of students that the extent of this course, as of all others, must be kept within narrow limits: and it has therefore been necessary to restrict the treatment to the most central and indispensable theorems. A large number of numerical illustrations and examples has been given of a kind likely to occur in the applications of Mathematics.

My thanks are due to Professor Whittaker and to my colleague, Mr E. M. Horsburgh, M.A., B.Sc., Assoc. M. Inst. C. E., for their valuable criticisms and suggestions.

D. G.

THE MATHEMATICAL LABORATORY,
UNIVERSITY OF EDINBURGH,
July 1915.

CONTENTS

PRELIMINARY NOTE ON COMPUTATION.

BEFORE introducing the formulae required in interpolation and numerical integration, it may be advisable to mention a few points which will facilitate the work of computing.

Where computation is performed to any considerable extent, computer's desks will be found useful. Those used in the mathematical laboratory of the University of Edinburgh are 3′ 0″ wide, 1′ 9″ from front to back, and 2′ $6\frac{1}{2}$″ high. They contain a locker, in which computing paper can be kept without being folded, and a cupboard for books, and are fitted with a strong adjustable book-rest. Thus the computer can command a large space and utilize it for books, papers, drawing-board, arithmometer, or instruments. Each desk is supplied with a copy of Barlow's tables (which give the square, square root, cube, cube root, and the reciprocal of all numbers up to 10,000), a copy of Crelle's multiplication table (which gives at sight the product of any two numbers each less than 1000), and with tables giving the values of the trigonometric functions and logarithms. These may, of course, be supplemented by a slide rule, or any of the various calculating machines* now in use, and such books of tables as bear particularly on the subject in question.

Success in computation depends partly on the proper choice of a formula and partly on a neat and methodical arrangement of the work. For the latter computing paper is essential. A convenient size for such a paper is 26″ by 16″; this should be divided by faint ruling into $\frac{1}{4}$″ squares, each of which is capable of

* For descriptions of these see *Modern Instruments and Methods of Calculation*, edited by E. M. Horsburgh. London: G. Bell & Sons. 1914.

holding two digits. It will be found conducive towards accuracy and speed if, instead of taking down a number one digit at a time, the computer takes it down two digits at a time, *e.g.*, instead of taking down the individual digits 2, 0, 4, 7, 6, 3, it will be found better to group them together, as 20, 47, 63. Every computation should be performed with ink in preference to pencil; this not only ensures a much more lasting record of the work but also prevents eye-strain and fatigue.

It need hardly be emphasized that seven-place accuracy cannot be obtained by the use of four-figure tables. A fairly safe rule is to make use of tables containing one digit more than the accuracy required; to use more accurate tables than these is simply to increase the amount of labour without increasing the efficiency. For the same reason contracted methods of multiplication and division should be adopted when machines and tables are not at hand.

CHAPTER I.

SOME THEOREMS IN THE CALCULUS OF FINITE DIFFERENCES.

1. Differences.

Let $\Delta f(x)$ denote the increment of a function $f(x)$ corresponding to a given constant increment w of the variable x, so that

$$\Delta f(x) = f(x+w) - f(x).$$

The expression $\Delta f(x)$, which is usually called the *First Difference* of the function $f(x)$ corresponding to the constant increment w of the variable x, will, in general, be another function of x, and as such will have a first difference which will be obtained by operating on $\Delta f(x)$ with Δ. Calling the result of this operation $\Delta^2 f(x)$, we have

$$\begin{aligned}\Delta^2 f(x) &= \Delta\,\{\Delta f(x)\}\\ &= \Delta\,\{f(x+w) - f(x)\}\\ &= \{f(x+2w) - f(x+w)\} - \{f(x+w) - f(x)\}\\ &= f(x+2w) - 2f(x+w) + f(x).\end{aligned}$$

This is termed the *Second Difference* of $f(x)$ corresponding to the constant increment w of x. By repeating the process we obtain in turn $\Delta^3 f(x)$, $\Delta^4 f(x)$, ... $\Delta^n f(x)$, which are called the 3rd, 4th, ... nth differences of $f(x)$ corresponding to this constant increment w.

Note.—It is always possible by means of a suitable transformation to make the value of w unity.

For since w is the increment of x in $f(x)$, then unity is the increment of $\dfrac{x}{w}$. Replacing x in $f(x)$ by wy, we obtain a new function $F(y)$, which is such that

$$\Delta F(y) = F(y+1) - F(y).$$

Example 1.—Tabulate and difference the successive values of $f(x)=x^3$ from $x=110$ to $x=118$.

x	x^3	Δ	Δ^2	Δ^3	Δ^4
110	1331000				
		36631			
111	1367631		666		
		37297		6	
112	1404928		672		0
		37969		6	
113	1442897		678		0
		38647		6	
114	1481544		684		0
		39331		6	
115	1520875		690		0
		40021		6	
116	1560896		696		0
		40717		6	
117	1601613		702		
		41419			
118	1643032				

In the above table the column headed x^3 was obtained from Barlow's Tables. The column Δ gives the first differences, being obtained by subtracting each entry of the column x^3 from the entry following it, *e.g.* if $x=114$, $x+1=115$, $\Delta f(x)=(x+1)^3-x^3=1520875-1481544=39331$. This result is set on a line midway between the numbers corresponding to $x=114$ and $x=115$. In the same way the 2nd differences, Δ^2, are obtained from the 1st differences, the results being set on a line midway between the latter, and so on.

It will be noticed that the 3rd differences of x^3 are constant and that its 4th differences are zero. It will be shown later that the nth differences of a function of the nth degree are constant and that the differences of higher order are all zero.

Example 2.—To determine the successive differences of $\sin(ax+b)$.

Let $f(x)=\sin(ax+b)$.

Then
$$\begin{aligned}
\Delta f(x) &= \sin(a\,\overline{x+w}+b)-\sin(ax+b)\\
&= 2\sin\tfrac12 aw\cos(ax+b+\tfrac12 aw)\\
&= 2\sin\tfrac12 aw\sin(ax+b+\tfrac12\,\overline{aw+\pi})\\
\Delta^2 f(x) &= \Delta\{2\sin\tfrac12 aw\sin(ax+b+\tfrac12\,\overline{aw+\pi})\}\\
&= 2\sin\tfrac12 aw\{\sin(a\,\overline{x+w}+b+\tfrac12\,\overline{aw+\pi})-\sin(ax+b+\tfrac12\,\overline{aw+\pi})\}\\
&= \{2\sin\tfrac12 aw\}^2\cos(ax+b+\tfrac12\,\overline{2aw+\pi})\\
&= \{2\sin\tfrac12 aw\}^2\sin(ax+b+\tfrac12\,\overline{2aw+2\pi}).
\end{aligned}$$

Similarly, it can be shown that

$$\Delta^3 f(x) = \{2\sin\tfrac{1}{2}a\,w\}^3 \sin(a\,x + b + \tfrac{1}{2}\overline{3\,a\,w + 3\,\pi})$$

and, more generally, that

$$\Delta^n f(x) = \{2\sin\tfrac{1}{2}a\,w\}^n \sin(a\,x + b + \tfrac{1}{2}\overline{n\,a\,w + n\,\pi}).$$

This can be proved by induction in the following way :—

$$\begin{aligned}\Delta^{n+1} f(x) &= \Delta\,[\{2\sin\tfrac{1}{2}a\,w\}^n \sin(a\,x + b + \tfrac{1}{2}\overline{n\,a\,w + n\,\pi})]\\ &= \{2\sin\tfrac{1}{2}a\,w\}^n\,[\sin(a\,\overline{x+w} + b + \tfrac{1}{2}\overline{n\,a\,w + n\,\pi})\\ &\qquad - \sin(a\,x + b + \tfrac{1}{2}\overline{n\,a\,w + n\,\pi})]\\ &= \{2\sin\tfrac{1}{2}a\,w\}^{n+1} \cos(a\,x + b + \tfrac{1}{2}\overline{(n+1)\,a\,w + n\,\pi})\\ &= \{2\sin\tfrac{1}{2}a\,w\}^{n+1} \sin(a\,x + b + \tfrac{1}{2}\overline{(n+1)\,a\,w + (n+1)\,\pi}).\end{aligned}$$

But the theorem has been proved for $n = 1, 2$. Hence it is true always.

Example 3.—Find the first differences of the functions

$$\cos(ax + b)\,;\ \tan^{-1} ax\,;\ a^{bx}.$$

Example 4.—Find the nth differences of the functions

$$\frac{1}{x}\,;\ a^x\,;\ \cos^3(ax + b).$$

2. Properties of the Operator Δ.

The operator Δ obeys certain of the fundamental laws of algebra, viz.—

(i) The *Law of Distribution.*

$$\begin{aligned}\text{For } \Delta\,\{f(x) + \phi(x)\} &= \{f(x+w) + \phi(x+w)\} - \{f(x) + \phi(x)\}\\ &= \{f(x+w) - f(x)\} + \{\phi(x+w) - \phi(x)\}\\ &= \Delta f(x) + \Delta\,\phi(x).\end{aligned}$$

(ii) The *Law of Commutation* if the coefficients of the various terms are constants.

For, if a is a constant,

$$\begin{aligned}\Delta\,\{a\,.f(x)\} &= a f(x+w) - a f(x)\\ &= a\{f(x+w) - f(x)\}\\ &= a\,.\,\Delta f(x).\end{aligned}$$

(iii) The *Law of Indices.*

$$\begin{aligned}&\text{For } \Delta^r\,.\,\Delta^s f(x)\\ &= \{\Delta\,.\,\Delta\,.\,\Delta \ldots (r \text{ factors})\,.\,\Delta\,.\,\Delta\,.\,\Delta \ldots (s \text{ factors})\} f(x)\\ &= \{\Delta\,.\,\Delta\,.\,\Delta \ldots \overline{(r+s} \text{ factors})\} f(x)\\ &= \Delta^{r+s} f(x).\end{aligned}$$

The indices r and s are, of course, integers: the operator Δ^n was defined in the previous section only for integral values of n.

Example 1.—If $f(x)$ is a polynomial in x of the nth degree, then its nth difference is constant and its differences of higher order than n are zero.

Let $\quad f(x)=a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

in which the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants.

Then
$$\begin{aligned}\Delta f(x) &= \{a_n (x+w)^n + a^{n-1}(x+w)^{n-1} + \ldots + a_1 (x+w) + a_0\} \\ &\quad - \{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\} \\ &= n\, a_n x^{n-1} w + b_{n-2} x^{n-2} + \ldots + b_1 x + b_0\end{aligned}$$

where $b_{n-2}, \ldots, b_1, b_0$ are constants.

We thus see that $\Delta f(x)$ is a polynomial in x of degree $(n-1)$. Repeating the operation, we have

$$\begin{aligned}\Delta^2 f(x) &= \{n\, a_n (x+w)^{n-1} w + b_{n-2}(x+w)^{n-2} + \ldots + b_1 (x+w) + b_0\} \\ &\quad - \{n\, a_n x^{n-1} w + b_{n-2} x^{n-2} + \ldots + b_1 x + b_0\} \\ &= n(n-1)\, a_n x^{n-2} w^2 + c_{n-3} x^{n-3} + \ldots + c_1 x + c_0\end{aligned}$$

where $c_{n-3}, \ldots, c_1, c_0$ are constants.

Hence $\Delta^2 f(x)$ is a polynomial in x of degree $(n-2)$. It is evident that by continued repetition of this process the degree of the function will be reduced to zero, and that we shall obtain

$$\Delta^n f(x) = n(n-1)(n-2) \ldots 3 \,.\, 2 \,.\, 1\, a_n w^n \,.$$

i.e., the nth difference of the function is a constant. Obviously the next and all differences of higher order will be zero. This result is of great importance ; probably no other theorem will be applied as frequently as this. For instance, it enables us to complete a mathematical table in which there are gaps. This is illustrated in the next example.

Example 2.—If

$$f(x)=0{\cdot}5563,\ f(x+1)=0{\cdot}5682,\ f(x+3)=0{\cdot}5911,\ f(x+5)=0{\cdot}6128,$$

obtain $f(x+2)$ and $f(x+4)$ on the assumption that $f(x)$ is a polynomial of the third degree.

It has been shown in § 1 that

$$\Delta^2 f(x) = f(x+2w) - 2f(x+w) + f(x).$$

By repeating the process we get

$$\Delta^4 f(x) = f(x+4w) - 4f(x+3w) + 6f(x+2w) - 4f(x+w) + f(x).$$

But the given function is of the third degree. Hence for it we have $\Delta^4 f(x)=0$. Noting that w is unity we obtain

$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

or, denoting $f(x+n)$ by f_n,

$$f_4 - 4f_3 + 6f_2 - 4f_1 + f_0 = 0.$$

Similarly $\qquad f_5 - 4f_4 + 6f_3 - 4f_2 + f_1 = 0.$

Substituting the given values in these two equations, we get

$$f_4 + 6f_2 = 4\cdot0809$$

$$f_4 + \ f_2 = 1\cdot1819$$

whence $$f_2 = 0\cdot5798\,, \qquad f_4 = 0\cdot6021\,,$$

i.e., $$f(x+2) = 0\cdot5798\,, \quad f(x+4) = 0\cdot6021.$$

Example 3.—If when $x = 0,\ 1,\ 3,\ 4,\ 6,$ the corresponding values of $f(x)$ are 1, 1, 79, 253, 1291, find approximate values for $f(x)$ when $x=2$ and when $x=5$. Under what conditions will the result be accurate?

3. Expansion of $f(x+nw)$ for Integral Values of n.

We shall now establish certain formulae which express $f(x+nw)$ (where n is an integer) in terms of differences of various orders. These formulae will be shown in Chapter II. to be valid under certain conditions even when n is not an integer.

It will be found convenient to introduce another operator E defined by the equation

$$E = 1 + \Delta.$$

This new operator will obviously obey the same laws as Δ.

Now $$Ef(x) = (1+\Delta)f(x) = f(x) + \Delta f(x) = f(x+w).$$

Thus the effect of the operator E on the function $f(x)$ is to increase its argument x by the given constant quantity w. Repeating the process we get

$$E^n f(x) = f(x+nw)$$

and since the operator Δ may under the conditions of § 2 be treated as an algebraical quantity, this last equation may be written

$$\begin{aligned} f(x+nw) &= (1+\Delta)^n f(x) \\ &= f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x) + \ \dots\ + \Delta^n f(x) \qquad (1) \end{aligned}$$

where ${}_nc_r$ denotes the coefficient of a^r in the binomial expansion of $(1+a)^n$. The function $f(x+nw)$ is here expressed in terms of $f(x)$ and its successive differences.

It is evident that, since the effect of the operator E on $f(x)$ is to increase x by w, the result of operating on $f(x)$ with E^{-1}, where E^{-1} is defined by the equation $EE^{-1} = 1$, will be to subtract w from x,

i.e., $$E^{-1}f(x) = f(x-w).$$

Hence the above formula (1) may be transformed as follows :

$$f(x+nw) = f(x) + {}_nc_1\,\Delta f(x) + (1+\Delta)\,E^{-1}\{{}_nc_2\,\Delta^2 f(x) + {}_nc_3\,\Delta^3 f(x) + \dots + \Delta^n f(x)\}$$

$$= f(x) + {}_nc_1\,\Delta f(x) + (1+\Delta)\{{}_nc_2\,\Delta^2 f(x-w) + {}_nc_3\,\Delta^3 f(x-w) + \dots + \Delta^n f(x-w)\}$$

$$= f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-w) + {}_{n+1}c_4\,\Delta^4 f(x-w) + \dots$$

$$= f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-w) + (1+\Delta)\,E^{-1}\{{}_{n+1}c_4\,\Delta^4 f(x-w) + \dots\}$$

$$= f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-w) + (1+\Delta)\{{}_{n+1}c_4\,\Delta^4 f(x-2w) + \dots\}$$

$$= f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-w) + {}_{n+1}c_4\,\Delta^4 f(x-2w) + {}_{n+2}c_5\,\Delta^5 f(x-2w) + \dots$$

Operating next with $(1+\Delta)\,E^{-1}$ on the 7th and subsequent terms, then on those beginning with the 9th term, and so on, we obtain

$$f(x+nw) = f(x) + {}_nc_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-w) + {}_{n+1}c_4\,\Delta^4 f(x-2w) + {}_{n+2}c_5\,\Delta^5 f(x-2w) + \dots \quad (2)$$

the formula ending at the term $\Delta^{2n-1} f(x - \overline{n-1}\,w)$.

In the same way it may be shown that

$$f(x+n\,w) = f(x) + {}_nc_1\,\Delta f(x-w) + {}_{n+1}c_2\,\Delta^2 f(x-w) + {}_{n+1}c_3\,\Delta^3 f(x-2w) + {}_{n+2}c_4\,\Delta^4 f(x-2w) + {}_{n+2}c_5\,\Delta^5 f(x-3w) + \dots \quad (3)$$

the formula ending at the term $\Delta^{2n} f(x - n\,w)$.

Changing x into $x+w$ and n into $n-1$ in (3) we have

$$f(x+n\,w) = f(x+w) + {}_{n-1}c_1\,\Delta f(x) + {}_nc_2\,\Delta^2 f(x) + {}_nc_3\,\Delta^3 f(x-w) + {}_{n+1}c_4\,\Delta^4 f(x-w) + {}_{n+1}c_5\,\Delta^5 f(x-2w) + \dots \quad (4)$$

the formula ending at the term $\Delta^{2n} f(x - \overline{n-2}\,w)$.

Adding (2) and (3) we obtain

$$f(x+nw)=f(x)+{}_nc_1\,\frac{\Delta f(x)+\Delta f(x-w)}{2}+\frac{n}{2}\,{}_nc_1\,\Delta^2 f(x-w)$$
$$+\,{}_{n+1}c_3\,\frac{\Delta^3 f(x-w)+\Delta^3 f(x-2w)}{2}$$
$$+\,\frac{n}{4}\,{}_{n+1}c_3\,\Delta^4 f(x-2w)+\ldots \qquad (5).$$

Similarly, addition of (2) and (4) gives

$$f(x+nw)=\frac{f(x+w)+f(x)}{2}+(n-\tfrac{1}{2})\,\Delta f(x)$$
$$+\,{}_nc_2\,\frac{\Delta^2 f(x)+\Delta^2 f(x-w)}{2}+\frac{n-\frac{1}{2}}{3}\,{}_nc_2\,\Delta^3 f(x-w)+\ldots \qquad (6).$$

Formulae (1), (2), (3), (4), (5), (6) may be compared with the expansions introduced in Chapter II. under the names Newton's, Stirling's, etc., Formulae of Interpolation.

Again, from the definition of the operator E, we see that

$$\Delta^n f(x)=(E-1)^n f(x).$$

Expanding the operator in this last relation we obtain

$$\Delta^n f(x)=E^n f(x)-{}_nc_1\,E^{n-1}f(x)+{}_nc_2\,E^{n-2}f(x)\ \ldots\ +(-1)^n f(x)$$
$$=f(x+nw)-{}_nc_1 f(x+\overline{n-1}\,w)+{}_nc_2 f(x+\overline{n-2}\,w)$$
$$+\ \ldots\ +(-1)^n f(x).$$

In this result we have the nth difference of the function $f(x)$ expressed explicitly in terms of $f(x)$ and its successive values.

4. Relations between the Differences and the Differential Coefficients of a Function.

The second example of § 1 may have reminded the reader of a somewhat similar result in the differential calculus, viz.,

$$\frac{d}{dx}\sin(ax+b)=a^n\sin(ax+b+\tfrac{1}{2}n\pi).$$

This result may be derived from the example in question. For if we put $w=\Delta x$, where Δx denotes a small increment in x, and,

after dividing both sides of the equation by $(\Delta x)^n$, make Δx tend to zero, we get

$$\frac{d^n f(x)}{dx^n} = \underset{\Delta x \to 0}{\mathrm{L}} \left\{\frac{2 \sin \frac{1}{2} a \,.\, \Delta x}{\Delta x}\right\}^n \sin \{a x + b + \tfrac{1}{2} n (a \,.\, \Delta x + \pi)\}.$$

$$= a^n \sin (a x + b + \tfrac{1}{2} n \pi).$$

This naturally suggests that there may be some relations connecting the successive differences of $f(x)$ and the successive differential coefficients of the same function. That such a relation does exist we now proceed to show.

Applying Taylor's Theorem* to the function $f(x+w)$ we obtain

$$f(x+w) = f(x) + w f'(x) + \frac{w^2}{2!} f''(x) + \ldots + \frac{w^m}{m!} f^{(m)}(x) + \ldots$$

or, since $f(x+w) - f(x) = \Delta f(x)$,

$$\Delta f(x) = w f'(x) + \frac{w^2}{2!} f''(x) + \ldots + \frac{w^m}{m!} f^{(m)}(x) + \ldots$$

which is a relation of the kind sought for.

Again, by Taylor's Theorem,

$$f(x+w) = f(x) + w f'(x) + \frac{w^2}{2!} f''(x) + \ldots + \frac{w^m}{m!} f^{(m)}(x) + \ldots$$

$$f(x+2w) = f(x) + 2w f'(x) + \frac{(2w)^2}{2!} f''(x) + \ldots + \frac{(2w)^m}{m!} f^{(m)}(x) + \ldots$$

and generally

$$f(x+nw) = f(x) + n w f'(x) + \frac{(nw)^2}{2!} f''(x) + \ldots + \frac{(nw)^m}{m!} f^{(m)}(x) + \ldots$$

Hence

$$\Delta^n f(x) = f(x + n w) - {}_nc_1 f(x + \overline{n-1}\, w) + {}_nc_2 f(x + \overline{n-2}\, w)$$
$$- \ldots + (-1)^n f(x)$$

$$= \Sigma \frac{w^m}{m!} \{n^m - {}_nc_1 (n-1)^m + {}_nc_2 (n-2)^m - \ldots\} f^{(m)}(x).$$

---

* It would be inopportune at this stage to introduce a discussion of the questioos of convergence which arise in connection with the application of Taylor's Theorem. We shall assume a convergence sufficient for our purpose.

But

$$n^m - {}_nc_1 (n-1)^m + {}_nc_2 (n-2)^m + \dots$$

$$= 0 \quad \text{if } m < n^*$$

$$= n! \quad \text{if } m = n$$

$$= \tfrac{1}{2} n (n+1)! \quad \text{if } m = n+1$$

$$= \frac{1}{4!} n (3n+1)(n+2)! \quad \text{if } m = n+2$$

etc.

Making use of these results in the expression for $\Delta^n f(x)$ we find that

$$\Delta^n f(x) = w^n f^{(n)}(x) + \frac{n}{2!} w^{n+1} f^{(n+1)}(x) + \frac{n(3n+1)}{4!} w^{n+2} f^{(n+2)}(x)$$
$$+ \frac{15 n^2 (n+1)}{6!} w^{n+3} f^{(n+3)}(x) + \dots$$

Corollary 1.—Since the nth differential coefficient of a function $f(x)$ of the nth degree is a constant, and the derivatives of higher order are zero, we see again from this expression for $\Delta^n f(x)$ that the nth difference of a polynomial of degree n is a constant, and the differences of higher order zero.

Corollary 2.—By reversing the series for $\Delta^n f(x)$, we obtain an expansion for the nth differential coefficient of $f(x)$ in terms of its differences, viz.

$$w^n f^{(n)}(x) = \Delta^n f(x) - \frac{n}{2!} \Delta^{n+1} f(x) + \frac{n(3n+5)}{4!} \Delta^{n+2} f(x)$$
$$- \frac{15 n (n+2)(n+3)}{6!} \Delta^{n+3} f(x) + \dots$$

Corollary 3.—If the nth differences of a function are constant, then the function is a polynomial of the nth degree.

For since the nth differences are constant, the higher differences are zero, and therefore the nth differential coefficient is constant,

---

* Chrystal's *Algebra*, Part II., Chap. XXVII., § 9.

and the differential coefficients of higher order zero. Thus the function, when expanded by Taylor's Theorem, takes the form

$$f(x) = f(X) + (x - X) f'(X) + \frac{(x - X)^2}{2!} f''(X) + \dots$$
$$+ \frac{(x - X)^n}{n!} f^{(n)}(X)$$

and this is of the nth degree in x.

Corollary 4.—Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$ we see that the series which represents $f(x+w)$, viz.

$$f(x) + wf'(x) + \frac{w^2}{2!} f''(x) + \dots$$

$$\text{or } \left\{1 + w\frac{d}{dx} + \frac{w^2}{2!}\left(\frac{d}{dx}\right)^2 + \dots + \frac{w^n}{n!}\left(\frac{d}{dx}\right)^n + \dots\right\} f(x)$$

may be symbolically represented as

$$e^{w\frac{d}{dx}} f(x).$$

Hence
$$E f(x) = e^{w\frac{d}{dx}} f(x).$$

MISCELLANEOUS EXAMPLES.

(Assume in the following exercises that w is unity, unless otherwise indicated).

1. Find the first differences of the functions

$$\tan ax; \quad \log x; \quad (a^{2x} - a^x)/(a-1).$$

2. Prove that

$$\Delta \tan x\theta = \frac{\sin \theta}{\cos x\theta \cos (x+1)\theta}.$$

3. Prove that

$$\Delta \frac{1}{2^x \tan \frac{\theta}{2^x}} = \frac{1}{2^{x+1}} \tan \frac{\theta}{2^{x+1}}.$$

4. Find the n^{th} differences of the functions

$$\frac{x+1}{x^2+3x}; \quad x^{m+n}; \quad \sin ax \cos bx.$$

5. Find the nth difference of each of the factorials $x(x+a)(x+2a)\dots$ and $\frac{1}{x(x+a)(x+2a)\dots}$, there being m factors in each factorial and the increment

of x being a: and show how to develop any polynomial in x in a series of factorials.

In particular show that

$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3).$$

6. If $f(x) = a\,3^x + b\,2^x$ prove that

$$\Delta^2 f(x) - 3\Delta f(x) + 2f(x) = 0.$$

7. Prove that

$$\Delta x^n - 2\Delta^2 x^n + 3\Delta^3 x^n - \ldots\ldots = (x-1)^n - (x-2)^n.$$

8. Establish the formula for the nth difference of the product of two functions

$$\Delta^n\{f(x)\,.\,\phi(x)\} = \Delta^n f(x)\,.\,\phi(x) + n\,\Delta^{n-1} f(x+1)\,.\,\Delta\,\phi(x) + \frac{n(n-1)}{2\,!}\Delta^{n-2} f(x+2)\,.\,\Delta^2\phi(x) + \ldots$$

9. Show that

$$\phi(E)\,a^x f(x) = a^x\,\phi(aE)\,f(x)$$

and find the value of

$$\Delta^2 f(x) + \frac{2}{a}(a - \cos b)\,\Delta f(x) + \left(1 - \frac{2}{a}\cos b + \frac{1}{a^2}\right) f(x)$$

where $$f(x) = \frac{\cos bx}{a^x}.$$

10. The record of exports for the year 1813 was destroyed by fire. Make an estimate of their value given that the values of the exports for the years 1810, 1811, 1812, 1814, 1815, 1816 were 48, 33, 42, 45, 52, and 42 million pounds respectively.

11. It is asserted that a quantity, which varies from day to day, is a rational and integral function of the day of the month, of less than the fifth degree, and that its values on the first seven days of the month are 30, 30, 28, 25, 22, 20, 20. Examine whether these assertions are consistent. If so, assume them to be true, and find (1) the degree of the function, (2) its value on the sixteenth day of the month.

CHAPTER II.

FORMULAE OF INTERPOLATION.

5. Introduction.

The problem of determining the value of a function for some intermediate value of the independent variable or *argument*, when the value of the function is given for a set of discrete values of the argument, is called *Interpolation*. When the function obeys some definite law the approximation to the true value may be made as accurate as may be desired: but if, as often happens, the rigorous analytical formulae which occur are very complicated, they will be of little value from the point of view of computation. It will then be necessary to use a method analogous to that which has to be employed when the properties of the function are entirely unknown, save for the values (usually obtained as a result of experiment or observation) which it has for a certain limited number of values of the argument. It is to this case that our attention will be directed.

When the observations are given for values of the argument equally distant from each other, and are in arithmetical progression so that they can be plotted on a straight line, the possibility of interpolating is obvious. But, in general, the observed values do not follow any apparent law, and unless some assumptions are made regarding the form of the function, the problem is indeterminate. It is, however, customary to obtain the entry for values of the argument not far distant from each other; and when this is done, in most practical problems it is found that all the differences at some particular order become sensibly zero. In such cases the observations may be represented with considerable accuracy by a polynomial in x of the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

This is the equation of a parabola of the nth order, on which account the method now to be described is sometimes called *Parabolic Interpolation*.

In some cases it is advisable to calculate the function not directly, but in combination with some other function. The calculation of the values of the probability integral $\int_x^\infty e^{-\frac{1}{2}t^2}\,dt$ furnishes an example of this kind, since, when the argument becomes large, the integral and its differences become inconveniently small and irregular. In this case it will be found convenient to compute the function $e^{\frac{1}{2}x^2}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt$. At other times some function of the given function is calculated; for example, if the observations formed a geometrical progression, then we should prefer to make the interpolation on their logarithms rather than on the quantities themselves.

Wallis (1616–1703) may be said to have laid down the principle of interpolation in his approximation to the area of the circle $y=(x-x^2)^{1/2}$. In this work he first assumed that the value of the integral $\int_0^1 (x-x^2)^{1/2}\,dx$ might be taken as the geometric mean of the values of

$$\int_0^1 (x-x^2)^0\,dx \text{ and } \int_0^1 (x-x^2)\,dx, \text{ i.e. of 1 and } \tfrac{1}{6}.$$

Subsequently he saw that this was not exactly true, and that the value of $\int_0^1 (x-x^2)^{1/2}\,dx$ must obey the law expressed by the series

$$\int_0^1 (x-x^2)^0\,dx, \quad \int_0^1 (x-x^2)\,dx, \ldots, \quad \int_0^1 (x-x^2)^n\,dx\,;$$

this was equivalent to interpolating the value of the integral under consideration.

6. Lagrange's Formula of Interpolation.

We shall first show how to find a polynomial of degree $(n-1)$ at most, which has the same values as some arbitrary function $f(x)$ for n distinct values of x, say $x=a_1, a_2, \ldots a_n$. This polynomial will be represented by the parabola of order $(n-1)$ already mentioned, and may be expressed in the form

$$f(x)=c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}.$$

But it will be found more convenient to write it, as is always possible, in the form

$$f(x)=\sum_{r=1}^{n} A_r(x-a_1)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n) \qquad (1)$$

When x is put equal to a_r all the terms on the right-hand side of (1) vanish except the term containing A_r: at the same time $f(x)$ becomes $f(a_r)$. Hence

$$A_r = f(a_r) \,/\, (a_r - a_1) \ldots (a_r - a_{r-1}) (a_r - a_{r+1}) \ldots (a_r - a_n)$$

and equation (1) becomes

$$f(x) = \sum_{r=1}^{n} \frac{(x - a_1) \ldots (x - a_{r-1}) (x - a_{r+1}) \ldots (x - a_n)}{(a_r - a_1) \ldots (a_r - a_{r-1}) (a_r - a_{r+1}) \ldots (a_r - a_n)} f(a_r) \qquad (2)$$

This formula expresses $f(x)$ for all values of x in terms of its values at the n points $a_1, a_2, \ldots a_n$. If $f(x)$ is a polynomial of degree less than n, the formula gives its accurate expression. But if $f(x)$ is not a polynomial of degree less than n, then it is no longer possible (in default of other information) to determine $f(x)$ from a knowledge of its values at the points $a_1, a_2, \ldots a_n$; for if any one such function were known we could obtain another by adding to it any function which is zero at the points $a_1, a_2, \ldots a_n$, *e.g.*, we could add the function $(x - a_1)(x - a_2) \ldots (x - a_n)\ \phi(x)$ where $\phi(x)$ does not become infinite when $x = a_1, a_2, \ldots a_n$. If, however, $f(x)$ is a function of which nothing is known except that it takes certain definite values corresponding to certain definite values $a_1, a_2, \ldots a_n$ of the variable, then the simplest assumption which we can make regarding its form is, that it is the function of degree $(n-1)$ which satisfies the given conditions; and this function is given by equation (2). The expression on the right-hand side of (2) may therefore be taken as the expression for $f(x)$ for all values of x in the interval considered. It may be written in the form

$$f(x) = \sum_{r=1}^{n} \frac{\phi(x)}{(x - a_r)\, \phi'(a_r)} f(a_r)$$

where $$\phi(x) = (x - a_1)(x - a_2) \ldots (x - a_n).$$

This is known as *Lagrange's Formula of Interpolation.* The advantage which it has over other formulae for the same purpose is that it is in a form suitable for logarithmic computation.

This latter form of Lagrange's formula may be obtained directly from the formula previously given. It can also be obtained very readily by the Method of Partial Fractions.

For consider the proper fractional function $\dfrac{f(x)}{\phi(x)}$ where

$$\phi(x) = (x - a_1)(x - a_2) \ldots (x - a_n).$$

Decomposing it into partial fractions we have

$$\frac{f(x)}{\phi(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\dots+\frac{A_n}{x-a_n}.$$

But, by the general theory of partial fractions, the coefficient A_r is given by

$$A_r=\frac{f(a_r)}{\phi'(a_r)}.$$

Hence

$$\frac{f(x)}{\phi(x)}=\frac{f(a_1)}{(x-a_1)\,\phi'(a_1)}+\frac{f(a_2)}{(x-a_2)\,\phi'(a_2)}+\dots+\frac{f(a_n)}{(x-a_n)\,\phi'(a_n)}$$

which gives

$$f(x)=\sum_{r=1}^{n}\frac{\phi(x)}{(x-a_r)\,\phi'(a_r)}\,f(a_r).$$

Example 1.—Construct a polynomial of the third degree in x which shall have the values -5, 1, 1, 7 when the values of x are -2, -1, 1, 2 respectively, and determine its value when x is $\frac{3}{2}$.

Let $f(x)$ be the required function. Substituting the given values in Lagrange's Interpolation Formula we have

$$\begin{aligned}
f(x) &= \frac{(x+1)(x-1)(x-2)}{(-2+1)(-2-1)(-2-2)}(-5)+\frac{(x+2)(x-1)(x-2)}{(-1+2)(-1-1)(-1-2)}(+1)\\
&\quad+\frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)}(+1)+\frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)}(+7)\\
&=\tfrac{5}{12}(x+1)(x-1)(x-2)+\tfrac{1}{6}(x+2)(x-1)(x-2)\\
&\qquad-\tfrac{1}{6}(x+2)(x+1)(x-2)+\tfrac{7}{12}(x+2)(x+1)(x-1)\\
&=x^3-x+1.
\end{aligned}$$

From this it follows that $f(\frac{3}{2})=2\frac{7}{8}$. But in practice it is not necessary to obtain the algebraic form of $f(x)$ if we merely require its value for some value of x. Direct substitution in the formula will give the result much more quickly. In the question under consideration

$$\begin{aligned}
f(\tfrac{3}{2}) &= \frac{(\frac{3}{2}+1)(\frac{3}{2}-1)(\frac{3}{2}-2)}{(-2+1)(-2-1)(-2-2)}(-5)+\frac{(\frac{3}{2}+2)(\frac{3}{2}-1)(\frac{3}{2}-2)}{(-1+2)(-1-1)(-1-2)}\\
&\quad+\frac{(\frac{3}{2}+2)(\frac{3}{2}+1)(\frac{3}{2}-2)}{(1+2)(1+1)(1-2)}+\frac{(\frac{3}{2}+2)(\frac{3}{2}+1)(\frac{3}{2}-1)}{(2+2)(2+1)(2+1)}(+7)\\
&=-\tfrac{25}{96}-\tfrac{7}{48}+\tfrac{35}{48}+\tfrac{245}{96}\\
&=2\tfrac{7}{8}.
\end{aligned}$$

Example 2.—Find the minimum value of the polynomial of the second degree which has the following values

x	0,	1,	5
$f(x)$	11,	5,	21.

Substituting these values in Lagrange's Interpolation Formula we have

$$f(x) = \frac{(x-1)(x-5)}{-1 \,.\, -5} \,.\, 11 + \frac{x(x-5)}{1 \,.\, -4} \,.\, 5 + \frac{x(x-1)}{5 \,.\, 4} \,.\, 21$$

$$\therefore \quad f'(x) = \tfrac{11}{5}(2x-6) - \tfrac{5}{4}(2x-5) + \tfrac{21}{20}(2x-1).$$

For a minimum turning value $f'(x)=0$

$$\begin{aligned} \text{i.e.,} \qquad 44(2x-6) - 25(2x-5) + 21(2x-1) &= 0 \\ 80x - 160 &= 0 \\ x &= 2 \end{aligned}$$

Hence the minimum value of $f(x)$ corresponds to the value *2* of the argument. The corresponding value of $f(x)$ is *3*.

Example 3.—If $f(x)$ is a function of x which assumes the values 20, 35, 56, 84, when x has the values 0, 1, 2, 3 respectively, express $f(x)$ in terms of x, on the supposition that $\Delta^4 f(x)=0$. Also find an approximate value of x if $f(x)=25$, third differences being neglected.

Example 4.—If four consecutive values of a function corresponding to the values 1, 2, 3, 4 of the variable be 601, 777, 902, 999 respectively, find the value of the function when the argument has the value 1·5.

7. Periodic Interpolation.

When there is reason to suppose that the function is periodic, it is better to assume for $f(x)$ an expansion in terms of periodic functions, say,

$$\begin{aligned} f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx, \end{aligned}$$

the number of unknown coefficients corresponding to the number of observations given. When the observations are given for values of the argument spaced at equal intervals, the problem becomes one in Fourier's analysis and will be found treated in the tract on that subject. But when the values of x are not in arithmetical progression we may use a formula which is analogous to Lagrange's formula of interpolation, and which Gauss has shown to be equivalent to the above expansion for $f(x)$.

If $f(x)$ is given for $x = x_0, x_1, x_2, \ldots x_{2n}$ the corresponding formula is

$$f(x) = \sum_{r=0}^{2n} \frac{\sin\frac{1}{2}(x-x_0)\ldots\sin\frac{1}{2}(x-x_{r-1})\sin\frac{1}{2}(x-x_{r+1})\ldots\sin\frac{1}{2}(x-x_{2n})}{\sin\frac{1}{2}(x_r-x_1)\ldots\sin\frac{1}{2}(x_r-x_{r-1})\sin\frac{1}{2}(x_r-x_{r+1})\ldots\sin\frac{1}{2}(x_r-x_{2n})} f(x_r).$$

8. Notation.

Hitherto the points $a_1, a_2, \ldots a_n$ have been taken to be any points whatever. But in the most important class of cases it is possible to obtain the entry for values of the argument equally distant from each other. In what follows we shall assume that the

arguments are spaced at equal intervals, and shall use the notation of the following scheme which for many purposes * is more convenient than that used in Chapter I.

Argument.	Entry.	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
$a-3w$	f_{-3}						
	$\mu f_{-\frac{5}{2}}$	$\delta f_{-\frac{5}{2}}$					
$a-2w$	f_{-2}	$\mu\delta f_{-2}$	$\delta^2 f_{-2}$				
	$\mu f_{-\frac{3}{2}}$	$\delta f_{-\frac{3}{2}}$	$\mu\delta^2 f_{-\frac{3}{2}}$	$\delta^3 f_{-\frac{3}{2}}$			
$a-w$	f_{-1}	$\mu\delta f_{-1}$	$\delta^2 f_{-1}$	$\mu\delta^3 f_{-1}$	$\delta^4 f_{-1}$		
	$\mu f_{-\frac{1}{2}}$	$\delta f_{-\frac{1}{2}}$	$\mu\delta^2 f_{-\frac{1}{2}}$	$\delta^3 f_{-\frac{1}{2}}$	$\mu\delta^4 f_{-\frac{1}{2}}$	$\delta^5 f_{-\frac{1}{2}}$	
a	f_0	$\mu\delta f_0$	$\delta^2 f_0$	$\mu\delta^3 f_0$	$\delta^4 f_0$	$\mu\delta^5 f_0$	$\delta^6 f_0$
	$\mu f_{\frac{1}{2}}$	$\delta f_{\frac{1}{2}}$	$\mu\delta^2 f_{\frac{1}{2}}$	$\delta^3 f_{\frac{1}{2}}$	$\mu\delta^4 f_{\frac{1}{2}}$	$\delta^5 f_{\frac{1}{2}}$	
$a+w$	f_1	$\mu\delta f_1$	$\delta^2 f_1$	$\mu\delta^3 f_1$	$\delta^4 f_1$		
	$\mu f_{\frac{3}{2}}$	$\delta f_{\frac{3}{2}}$	$\mu\delta^2 f_{\frac{3}{2}}$	$\delta^3 f_{\frac{3}{2}}$			
$a+2w$	f_2	$\mu\delta f_2$	$\delta^2 f_2$				
	$\mu f_{\frac{5}{2}}$	$\delta f_{\frac{5}{2}}$					
$a+3w$	f_3						

where $\delta f_{-\frac{1}{2}} = f_0 - f_{-1}$ $\qquad \mu f_{\frac{1}{2}} = \frac{1}{2}(f_0 + f_1)$

$\delta f_{\frac{1}{2}} = f_1 - f_0$ $\qquad \mu\delta f_0 = \frac{1}{2}(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}})$

etc. etc.

9. Newton's Formula of Interpolation.

If we assume that the differences $a_2 - a_1$, $a_3 - a_2$, ... in Lagrange's interpolation formula are all equal to w, and that a_1 in the formula corresponds to a in the scheme, then the quantities $f(a_r)$ take the following forms:—

$$f(a_1) = f_0$$

$$f(a_2) = f_1 = f_0 + \delta f_{\frac{1}{2}}$$

$$f(a_3) = f_2 = f_1 + \delta f_{\frac{3}{2}} = f_0 + 2\,\delta f_{\frac{1}{2}} + \delta^2 f_1$$

....

$$f(a_{r+1}) = f_0 + r\,\delta f_{\frac{1}{2}} + \frac{r(r-1)}{2!}\delta^2 f_1 + \frac{r(r-1)(r-2)}{3!}\delta^3 f_{\frac{3}{2}} + \dots$$

* Especially in connexion with central-difference formulae.

Lagrange's Interpolation Formula then becomes, if the number of given points is denoted by k,

$$f(x)=\frac{(x-a_2)(x-a_3)(x-a_4)\dots(x-a_k)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)\dots(a_1-a_k)}f_0$$

$$+\frac{(x-a_1)(x-a_3)(x-a_4)\dots(x-a_k)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)\dots(a_2-a_k)}(f_0+\delta f_{\frac{1}{2}})$$

$$+\frac{(x-a_1)(x-a_2)(x-a_4)\dots(x-a_k)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)\dots(a_3-a_k)}(f_0+2\,\delta f_{\frac{1}{2}}+\delta^2 f_1)$$

$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\frac{(x-a_1)\dots(x-a_r)(x-a_{r+2})\dots(x-a_k)}{(a_{r+1}-a_1)\dots(a_{r+1}-a_r)(a_{r+1}-a_{r+2})\dots(a_{r+1}-a_k)}$$

$$\times\left(f_0+r\,\delta f_{\frac{1}{2}}+\frac{r(r-1)}{1\,.\,2}\,\delta^2 f_1+\dots\right)$$

$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where $a_1=a$, $a_2=a+w$, ..., $a_k=a+(k-1)w$.

This is an expression of degree $(k-1)$ in x. In it the coefficient of f_0 is unity when $x=a_1, a_2, \dots a_k$, and hence is unity always. Again, the coefficient of $\delta f_{\frac{1}{2}}$ is 0 at a_1, 1 at a_2, 2 at a_3, ... r at a_{r+1}, ..., and as this coefficient must be a polynomial in x of degree less than k, it must be no other than $\frac{x-a}{w}$. In fact the terms in $\delta f_{\frac{1}{2}}$ in the above series simply constitute the expansion of $\frac{x-a}{w}\delta f_{\frac{1}{2}}$ by Lagrange's formula. We shall denote $\frac{x-a}{w}$ by n, so that n is a real quantity, not necessarily an integer. The coefficient of $\delta f_{\frac{1}{2}}$ is thus n. Similarly the coefficient of $\delta^2 f_1$ is 0 at a_1, 0 at a_2, 1 at a_3, ... $\frac{r(r-1)}{1\,.\,2}$ at a_{r+1} ..., and in the same way we see that this coefficient is $\frac{n(n-1)}{2\,!}$ for any value of n. The other coefficients can be determined in the same way, and we thus finally have Lagrange's formula transformed into the equation

$$f(a+nw)=f_0+n\,\delta f_{\frac{1}{2}}+\frac{n(n-1)}{2\,!}\,\delta^2 f_1+\frac{n(n-1)(n-2)}{3\,!}\,\delta^3 f_{\frac{3}{2}}+\dots$$

This is called *Newton's Formula of Interpolation.** In it the differences are those which occur along a line parallel to the upper side of the triangle in the scheme. It may on this account be termed a *Forward Difference* formula of interpolation, and will be found very useful when the value of the function near the beginning of a limited series of observations is required.

It must be understood that the function furnished by Newton's formula is the polynomial of lowest degree which has the given values at the k given points a, $a+w$, $a+2w$, ..., $a+(k-1)w$: and the series on the right is a *terminating* series. If the given values at the k given points are the values of some given function $\phi(x)$ at the points, we cannot conclude that $\phi(x)$ is represented by Newton's formula unless $\phi(x)$ is the polynomial of lowest degree which passes through the points.

Suppose, now, that the values of a function are given not merely for a finite set of values of its argument, but for the infinite set

$$x = a + rw \qquad (r = 0, 1, 2, 3, \ldots \textit{ ad inf.}).$$

Then we can, by Newton's formula, construct a polynomial of degree $(k-1)$ which has the given values for the first k of these values of the argument, where k is any positive integer: and it may happen that, as k increases indefinitely, the series on the right in Newton's formula becomes (for some range of values of x) a convergent series, having for its sum a function $f(x)$. The function $f(x)$ will now have the given values for each of the values $a+rw$ ($r = 0, 1, 2, 3, \ldots$ *ad inf.*) of the argument. There are, however, an infinite number of functions which satisfy this condition and $f(x)$ is that one of them which is the limit, when $k \to \infty$, of the polynomial of lowest degree which has k of the given values.

An expression for the difference between the value of the function and the sum of the first $(r+1)$ terms of the limiting form of this polynomial will now be obtained. Let

$$f(a+nw) = f_0 + n\,\delta f_{\frac{1}{2}} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}\,\delta^r f_{\frac{r}{2}} + R.$$

Then

$$f(a+mw) - f_0 - m\,\delta f_{\frac{1}{2}} - \ldots - \frac{m(m-1)\ldots(m-r+1)}{r!}\,\delta^r f_{\frac{r}{2}} - R$$

* *Cf.* § 3, Formula (1).

will be zero for $m=0, 1, 2, \ldots r$ and n: its derivate of order $(r+1)$ will therefore be zero for some value n' of m lying between the least and the greatest of the quantities 0, r, n, and hence the residue is given by

$$R = {}_nC_{r+1}\, w^{r+1} f^{(r+1)}(a + n'w).$$

It may be remarked that Newton discovered the binomial coefficients when studying the expansion in series of a function by interpolations. He first of all considered the expansions of the expressions $(1-x^2)^0$, $(1-x^2)^1$, $(1-x^2)^2$, ... and deduced from them the expansions of $(1-x^2)^{\frac{1}{2}}$, $(1-x^2)^{\frac{3}{2}}$, ... Then, by analogy, he obtained the expression for the general term in the expansion of a binomial, and thus established the binomial theorem.

Example 1.—From the following table of values of the function $f(x)$ determine the value of $f(3{\cdot}5)$.

x	$f(x)$	Δ	Δ^2	Δ^3
1	0·208460			
		29242		
2	0·237702		−213	
		29029		−27
3	0·266731		−240	
		28789		−26
4	0·295520		−266	
		28523		
5	0·324043			

In this example $\quad a=1, \quad n=\frac{1}{2}, \quad w=1$

$$f_0 = f(1) = 0\ 208460.$$

Hence
$$f(3{\cdot}5) = f_0 + \tfrac{1}{2}\,\delta f_{\frac{1}{2}} \quad \tfrac{1}{8}\,\delta^2 f_1 + \tfrac{1}{16}\,\delta^3 f_{\frac{5}{2}}.$$
$$= 0{\cdot}208460$$
$$14621$$
$$27 - 2$$
$$= 0{\cdot}223106.$$

Example 2.—Corresponding values of x and $f(x)$ are given in the following table :—

x	$f(x)$
4·0	54·5982
4·1	60·3403
4·2	66·6863
4·3	73·6998
4·4	81·4509
4·5	90·0171

Obtain the value of $f(4{\cdot}05)$.

A certain amount of discretion is necessary in the application of Newton's formula. Let us assume, for instance, that the given observations are finite — say seven — in number, and correspond to the values $a-3w$, $a-2w$, ... $a+3w$ of the argument as in the scheme, § 8. If $a+nw$ lies between $a-3w$ and $a-2w$, then Newton's formula for $f(a+nw)$ will involve the quantities f_{-3}, $\delta f_{-\frac{5}{2}}$, $\delta^2 f_{-2}$, $\delta^3 f_{-\frac{3}{2}}$, $\delta^4 f_{-1}$, $\delta^5 f_{-\frac{1}{2}}$ and $\delta^6 f_0$, all of which occur in the scheme. Thus a good approximation to the value of $f(a+nw)$ may be obtained by means of the formula. But if $a+nw$ lies between $a+2w$ and $a+3w$, then in the Newton's formula for $f(a+nw)$ the only known terms are the first two depending on f_2 and $\delta f_{\frac{5}{2}}$. In general, these are quite insufficient to determine the value of $f(a+nw)$ to the necessary degree of accuracy. It is therefore expedient in this case to find a formula involving a larger number of known quantities. A suitable one may be obtained by replacing w in the original formula by $-w$. Then, since

$$\delta f_{\frac{1}{2}} = f_0 - f_1,$$

we must replace $\delta f_{\frac{1}{2}}$ by

$$f_0 - f_{-1} \text{ or } -\delta f_{-\frac{1}{2}}.$$

Similarly, since

$$\delta^2 f_1 = f_0 - 2f_1 + f_2,$$

we must replace $\delta^2 f_1$ by

$$f_0 - 2f_{-1} + f_{-2} \text{ or } \delta^2 f_{-1}.$$

In the same way we can determine the functions which must take the places of $\delta^3 f_{\frac{3}{2}}$, $\delta^4 f_4$, ... Newton's formula then becomes

$$f(a-nw) = f_0 - n\,\delta f_{-\frac{1}{2}} + \frac{n(n-1)}{2\,!}\,\delta^2 f_{-1} - \frac{n(n-1)(n-2)}{3\,!}\,\delta^3 f_{-\frac{3}{2}} + \dots$$

This expansion involves differences along a line parallel to the lower side in the scheme, and it may therefore be regarded as a formula for *Backward Interpolation*. When $a+nw$ lies between $a+2w$ and $a+3w$, the formula will depend on the quantities f_3, $\delta f_{\frac{5}{2}}$, $\delta^2 f_2$, $\delta^3 f_{\frac{3}{2}}$, $\delta^4 f_1$, $\delta^5 f_{\frac{1}{2}}$, and $\delta^6 f_0$, all of which are known: it is therefore very convenient when the value of $f(x)$ is required for values of x near the end of the series of observations.

Example 3.—From the following table of values of $f(x)$ determine the values of $f(104 \cdot 25)$.

x	$f(x)$	Δ	Δ^2	Δ^3
101	1030198			
		30906		
102	1061104		612	
		31518		6
103	1092622		618	
		32136		6
104	1124758		624	
		32760		
105	1157518			

In this example $a = 105, \quad n = 0 \cdot 75, \quad w = 1$

$$f_0 = f(105) = 1157518$$

$f(104 \cdot 25)$

$$= f_0 - 0 \cdot 75\, \delta f_{-\frac{1}{2}} + \frac{0 \cdot 75\,(0 \cdot 75 - 1)}{2\,!}\, \delta^2 f_{-1} - \frac{0 \cdot 75\,(0 \cdot 75 - 1)\,(0 \cdot 75 - 2)}{3\,!}\, \delta^3 f_{-\frac{3}{2}} + \ldots$$

$$= f_0 - 0 \cdot 75\, \delta f_{-\frac{1}{2}} - 0 \cdot 09375\, \delta^2 f_{-1} - 0 \cdot 0390625\, \delta^3 f_{\frac{3}{2}} + \ldots$$

$$= 1157518 - 0 \cdot 75\,(32760) - 0 \cdot 09375\,(624) - 0 \cdot 0390625\,(6)$$

$$= 1157518 - 24570$$
$$58 \cdot 5$$
$$0 \cdot 234375$$

$$= 1132889 \cdot 265625.$$

Example 4.—Using the data of Example 2, find the value of $f(4 \cdot 45)$.

10. Stirling's Formula of Interpolation.

As Newton's formula involves differences which lie along a slanting line it will frequently not be suitable for calculating the value of the function for a value of the argument near the middle of a series of observations. By a suitable transformation of this formula, however, we may derive another involving differences which lie on a horizontal line; such a formula is termed a *Central-Difference Formula.*

Since

$$\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}} = 2\,\mu\,\delta f_0$$
$$\delta f_{\frac{1}{2}} - \delta f_{-\frac{1}{2}} = \delta^2 f_0$$

we have

$$\delta f_{\frac{1}{2}} = \mu\,\delta f_0 + \tfrac{1}{2}\,\delta^2 f_0 .$$

Hence

$$\delta^2 f_1 - \delta^2 f_0 = \delta^3 f_{\frac{1}{2}}$$
$$= \mu\,\delta^3 f_0 + \tfrac{1}{2}\,\delta^4 f_0$$
$$\therefore \quad \delta^2 f_1 = \delta^2 f_0 + \mu\,\delta^3 f_0 + \tfrac{1}{2}\,\delta^4 f_0.$$

In the same way the higher differences which occur in Newton's formula may be determined in terms of the differences which occur on the same horizontal line as f_0. Substituting these in Newton's formula we obtain

$$f(a+nw) = f_0 + n\{\mu\delta f_0 + \tfrac{1}{2}\delta^2 f_0\} + \frac{n(n-1)}{2!}\{\delta^2 f_0 + \mu\delta^3 f_0 + \tfrac{1}{2}\delta^4 f_0\}$$

$$+ \frac{n(n-1)(n-2)}{3!}\{\mu\delta^3 f_0 + \tfrac{3}{2}\delta^4 f_0 + \mu\delta^5 f_0 + \tfrac{1}{2}\delta^6 f_0\} + \dots$$

$$= f_0 + n\mu\delta f_0 + \frac{n^2}{2!}\delta^2 f_0 + \frac{n(n^2-1)}{3!}\mu\delta^3 f_0 + \frac{n^2(n^2-1)}{4!}\delta^4 f_0$$

$$+ \frac{n(n^2-1)(n^2-4)}{5!}\mu\delta^5 f_0 + \dots$$

This is known as *Stirling's Formula of Interpolation*. It should not be used for determining the value of a function for values of the argument near the beginning or the end of a series of observations, as it would not then be possible to obtain a sufficiently high order of differences.

Example 1.—Using the data of § 9, Example 1, determine the value of $f(3{\cdot}5)$.

x	$f(x)$	Δ	Δ^2	Δ^3
1	0·208460			
		29242		
2	0·237702		−213	
		29029		−27
3	0·266731	(28909)	−240	(−27)
		28789		−26
4	0·295520	(28656)	−266	(−26)
		28523		−26
5	0·324043		−292	
		28231		
6	0·352274			

In this case $a = 3$, $w = 1$, $n = \frac{1}{2}$

$$f_0 = f(3) = 0{\cdot}266731.$$

$$f(3{\cdot}5) = f_0 + \tfrac{1}{2}\mu\delta f_0 + \tfrac{1}{8}\delta^2 f_0 - \tfrac{1}{16}\delta^3 f_0$$

$$= 0{\cdot}266731$$
$$14455$$
$$2-30$$
$$= 0{\cdot}281158.$$

Example 2.—Using the data of § 9, Example 2, find the value of $f(4{\cdot}21)$.

If n is greater than 0·5, it is better to use backward differences. The corresponding formula may be obtained by merely changing the sign of w in Stirling's Forward Difference Formula, viz. :—

$$f(a-nw)=f_0-n\mu\,\delta f_0+\frac{n^2}{2\,!}\,\delta^2 f_0-\frac{n(n^2-1)}{3\,!}\,\mu\,\delta^3 f_0$$
$$+\frac{n^2(n^2-1)}{4\,!}\,\delta^4 f_0-\frac{n(n^2-1)(n^2-4)}{5\,!}\,\mu\,\delta^5 f_0+\dots$$

Of course, either formula may be applied whether n is greater or less than 0·5: thus, when a check is necessary, it will be found better to compute $f(x)$ by the formula not already used.

Example 3.—Use this *Backward Difference Formula* to check Example 1.

Here $\quad a=4,\quad w=1,\quad n=\frac{1}{2}$

$$f_0=f(4)=0{\cdot}295520$$
$$f(3{\cdot}5)=f_0-\tfrac{1}{2}\mu\,\delta f_0+\tfrac{1}{8}\delta^2 f_0+\tfrac{1}{16}\delta^3 f_0$$
$$=0{\cdot}295520-14328$$
$$33$$
$$2$$
$$=0{\cdot}281157.$$

The agreement with the former result is perfect when we take into consideration the fact that the last figure is forced in both cases.

Example 4.—Using the data of § 9, Example 2, find the value of $f(4{\cdot}29)$.

11. Gauss' and Bessel's Formulae of Interpolation.

Another central difference formula may be obtained by transforming Stirling's formula.

Since
$$\mu\,\delta f_0=\delta f_{\frac{1}{2}}-\tfrac{1}{2}\,\delta^2 f_0$$
$$\mu\,\delta^3 f_0=\delta^3 f_{\frac{1}{2}}-\tfrac{1}{2}\,\delta^4 f_0 \qquad \text{etc.,}$$

we see that Stirling's formula may be written in the form

$$f(a+nw)=f_0+n\,\delta f_{\frac{1}{2}}+\frac{n(n-1)}{2\,!}\,\delta^2 f_0+\frac{(n+1)\,n\,(n-1)}{3\,!}\,\delta^3 f_{\frac{1}{2}}$$
$$+\frac{(n+1)\,n\,(n-1)(n-2)}{4\,!}\,\delta^4 f_0$$
$$+\frac{(n+2)(n+1)\,n\,(n-1)(n-2)}{5\,!}\,\delta^5 f_{\frac{1}{2}}+\dots$$

In this result, which is frequently known as *Gauss' Formula*, the μ-terms do not occur. The differences which occur in this

formula are those which lie on two adjacent horizontal lines of the scheme in § 8.

Again, since

$$f_1 + f_0 = 2\,\mu f_{\frac{1}{2}}$$
$$f_1 - f_0 = \delta f_{\frac{1}{2}}$$

we have

$$f_0 = \mu f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta f_{\frac{1}{2}}$$
$$\delta^2 f_0 = \mu\,\delta^2 f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta^3 f_{\frac{1}{2}} \qquad \text{etc.,}$$

and hence

$$\begin{aligned} f(a+nw) &= \mu f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta f_{\frac{1}{2}} + n\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2\,!}\{\mu\,\delta^2 f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta^3 f_{\frac{1}{2}}\} + \frac{(n+1)n(n-1)}{3\,!}\,\delta^3 f_{\frac{1}{2}} \\ &\quad + \frac{(n+1)\,n\,(n-1)\,(n-2)}{4\,!}\,\{\mu\,\delta^4 f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta^5 f_{\frac{1}{2}}\} \\ &\quad + \frac{(n+2)\,(n+1)\,n\,(n-1)\,(n-2)}{5\,!}\,\delta^5 f_{\frac{1}{2}} + \dots \\ &= \mu f_{\frac{1}{2}} + (n-\tfrac{1}{2})\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2\,!}\,\mu\,\delta^2 f_{\frac{1}{2}} + \frac{n\,(n-1)\,(n-\tfrac{1}{2})}{3\,!}\,\delta^3 f_{\frac{1}{2}} \\ &\quad + \frac{(n+1)\,n\,(n-1)\,(n-2)}{4\,!}\,\mu\,\delta^4 f_{\frac{1}{2}} \\ &\quad + \frac{(n+1)\,n\,(n-1)\,(n-2)\,(n-\tfrac{1}{2})}{5\,!}\,\delta^5 f_{\frac{1}{2}} + \dots \end{aligned}$$

This is frequently known as *Bessel's Formula*. In it the differences are those which occur on a horizontal line mid-way between the lines on which f_0 and f_1 stand.

The formula most suitable for the construction of mathematical tables, when differences of higher order than the fourth are neglected, is obtained from the first of these on putting

$$\delta^4 f_0 = \mu\,\delta^4 f_{\frac{1}{2}} - \tfrac{1}{2}\,\delta^5 f_{\frac{1}{2}}\,.$$

When this substitution is made we get

$$\begin{aligned} f(a+nw) &= f_0 + n\,\delta f_{\frac{1}{2}} + \frac{n(1-n)}{2\,!}\,\delta^2 f_0 - \frac{n(1-n^2)}{3\,!}\,\delta^3 f_{\frac{1}{2}} \\ &\quad + \frac{n\,(1-n^2)\,(2-n)}{4\,!}\,\mu\,\delta^4 f_{\frac{1}{2}}\,. \end{aligned}$$

Bessel's formula will be found particularly suitable when n is equal to $\frac{1}{2}$, for in this case the formula becomes

$$f(a+\tfrac{1}{2}w)=\mu f_{\frac{1}{2}}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2\,!}\,\mu\,\delta^2 f_{\frac{1}{2}}+\frac{(\frac{1}{2}+1)\,\frac{1}{2}\,(\frac{1}{2}-1)\,(\frac{1}{2}-2)}{4\,!}\,\mu\,\delta^4 f_{\frac{1}{2}}+\dots$$

in which only even differences occur.

As with Stirling's formula, none of these results should be employed when the value of the function is required for values of the argument near the beginning or the end of a series of observations. Stirling's formula will be found more convenient when n is very small, or when the number of observations given is odd; Bessel's when n is nearly equal to a half, or the number of observations even.

The backward difference formula corresponding to Bessel's forward difference formula may be obtained in the same way as in Newton's and Stirling's cases. But, except as regards sign, the coefficients in the two formulae will be the same, and therefore the one cannot be used as a check on the other. When a check is required to a result computed from Bessel's formula, it is better to compute it again using Stirling's formula.

Example 1.—The values of a function for the values -2, -1, 0, 1, 2, of the argument are respectively 0·12569, 0·17882, 0·23004, 0·27974, 0·32823. Using Gauss' formula, show that when the argument has the value 0·4, the value of the function is 0·25008.

x	$f(x)$	Δ	Δ^2	Δ^3
-2	0·12569			
		5313		
-1	0·17882		-191	
		5122		39
0	0·23004		-152	
		4970		31
1	0·27974		-121	
		4849		
2	0·32823			

In this example $a=0,\quad n=0{\cdot}4,\quad w=1$

$$f_0=f(0)=0{\cdot}23004.$$

Hence
$$\begin{aligned} f(0{\cdot}4) &= f_0+0{\cdot}4\,\delta f_{\frac{1}{2}}-0{\cdot}12\,\delta^2 f_0 \quad 0{\cdot}056\,\delta^3 f_{\frac{1}{2}} \\ &= 0{\cdot}23004 \\ &\qquad 1988 \\ &\qquad\quad 18-2 \\ &= 0{\cdot}25008. \end{aligned}$$

Example 2.—Using the data of § 9, Example 2, find the values of $f(4\cdot21)$ and $f(4\cdot29)$.

12. Evaluation of the Derivatives of the Function.

When, as often happens, the values of the differential coefficients of a function which is given by a series of observations are required, use may be made of the symbolic equality established in § 4, Cor. 4.

For, since

$$e^{w\frac{d}{dx}} f(x) = Ef(x)$$

we have

$$w\frac{d}{dx}\,.\,f(x) = \log E\,.\,f(x).$$

The method will be evident from the following example:—

Example 1.—The values of a function corresponding to the values 1·01, 1·02, 1·03, 1·04, 1·05 of the argument are 1·030301, 1·061208, 1·092727, 1·124864, 1·157625 respectively. Find the value of the first differential coefficient of the function when the argument has the value 1·03.

Here $w = 0\cdot01$. Let $f_0 = f(1\cdot01)$.

Then
$$\begin{aligned} w\frac{d}{dx}f(1\cdot03) &= \log E\,.\,f(1\cdot03) \\ &= \log E\,.\,f_2 \\ &= \log E\,.\,E^2 f_0 \\ &= \log(1+\Delta)\,.\,(1+\Delta)^2.\,f_0 \\ &= \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots\right).\,(1+2\,\Delta+\Delta^2)\,.\,f_0. \end{aligned}$$

Neglecting differences of higher order than the 4th, this gives

$$\begin{aligned} 0\cdot01\,\frac{d}{dx}\,f(1\cdot03) &= (\Delta + \tfrac{3}{2}\,\Delta^2 + \tfrac{1}{3}\,\Delta^3 - \tfrac{1}{12}\,\Delta^4) f_0 \\ &= (\tfrac{1}{12} - \tfrac{2}{3}\,E + \tfrac{2}{3}\,E^3 - \tfrac{1}{12}\,E^4) f_0 \\ &= \tfrac{2}{3}\,(f_3 - f_1) - \tfrac{1}{12}\,(f_4 - f_0) \\ &= 0\cdot042437 - 0\cdot010610 \\ &= 0\cdot031827 \\ \therefore\quad f'(1\cdot03) &= 3\cdot1827. \end{aligned}$$

Example 2.—The values of a function corresponding to the values 1·5, 2, 2·5, 3, 3·5, 4, of the argument are 4·625, 2·000, $-0\cdot125$, -1, 0·125, 4·000, respectively. Find the value of the first differential coefficient of the function at the points where the argument has the values 2·5 and 3·0.

The method is perfectly general. The same result may be obtained by differentiating any of the formulae of interpolation already established. For example, if we differentiate with respect to n the formula of Stirling,

$$f(a+nw) = f_0 + n\mu\delta f_0 + \frac{n^2}{2!}\delta^2 f_0 + \frac{n(n^2-1)}{3!}\mu\delta^3 f_0 + \frac{n^2(n^2-1)}{4!}\delta^4 f_0 + \dots$$

we have

$$wf'(a+nw) = \mu\delta f_0 + n\delta^2 f_0 + \tfrac{1}{6}(3n^2-1)\mu\delta^3 f_0 + \tfrac{1}{12}(2n^3-n)\delta^4 f_0 + \dots$$

$$w^2 f''(a+nw) = \delta^2 f_0 + n\mu\delta^3 f_0 + \tfrac{1}{12}(6n^2-1)\delta^4 f_0 + \dots$$

....

Putting $n=0$ these become

$$wf'(a) = \mu\delta f_0 - \tfrac{1}{6}\mu\delta^3 f_0 + \tfrac{1}{30}\mu\delta^5 f_0 - \dots$$

$$w^2 f''(a) = \delta^2 f_0 - \tfrac{1}{12}\delta^4 f_0 + \tfrac{1}{90}\delta^6 f_0 - \dots$$

....

These formulae are of use in determining the turning values of a function, the values of the argument corresponding to given values of the differential coefficients, the points of inflexion on a curve of which isolated points are given, and also in determining whether a series of observations is periodic.

Example 3.—The consecutive daily observations of a function being 0·099833, 0·208460, 0·314566, 0·416871, 0·514136, 0·605186, 0·688921, 0·764329, show that the function is periodic and determine its period.

From the given observations we obtain the table :—

x	$y=f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	0·099833				
		108627			
2	0·208460		−2521		
		106106		−1280	
3	0·314566		−3801		41
		102305		−1239	
4	0·416871		−5040		64
		97265		−1175	
5	0·514136		−6215		75
		91050		−1100	
6	0·605186		−7315		88
		83735		−1012	
7	0·688921		−8327		
		75408			
8	0·764329				

Taking $\qquad a = 3, \quad f_0 = f(a) = f(3),$ we have, since $w = 1$,

$$\begin{aligned} f''(3) &= \delta^2 f_0 - \tfrac{1}{12}\delta^4 f_0 \\ &= -3801 - \tfrac{1}{12}\cdot 41 \\ &= -3804 \end{aligned}$$

$$\therefore \quad \frac{1}{f(3)}\cdot f''(3) = -\frac{3804}{314566}$$

$$= -0{\cdot}0121.$$

In the same way we find that

(i) When $\quad a = 4, \quad f_0 = f(4)$

$$f''(4) = -5045, \quad \frac{1}{f(4)} f''(4) = -0{\cdot}0121$$

(ii) When $\quad a = 5, \quad f_0 = f(5)$

$$f''(5) = -6221, \quad \frac{1}{f(5)} f''(5) = -0{\cdot}0121$$

(iii) When $\quad a = 6, \quad f_0 = f(6)$

$$f''(6) = -7322, \quad \frac{1}{f(6)} f''(6) = -0{\cdot}0121.$$

Hence in all cases $\frac{1}{f(x)} f''(x) = -0{\cdot}0121$. Putting $y = f(x)$, this gives the differential equation

$$\frac{1}{y}\,\frac{d^2y}{dx^2} = -0{\cdot}0121$$

whence

$$\frac{d^2y}{dx^2} + 0{\cdot}0121\,y = 0$$

$$y = A\cos 0{\cdot}11\,x + B\sin 0{\cdot}11\,x.$$

This result shows that y or $f(x)$ is a periodic function of x, and that its period is $2\pi/0{\cdot}11$, or 57·12 days.

Example 4.—The following table represents one of the Bessel functions, $J_n(x)$:—

x	y
1·0	0·765198
1·1	0·719622
1·2	0·671133
1·3	0·620086
1·4	0·566855
1·5	0·511828
1·6	0·455402

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1{\cdot}3$, and hence, by substituting (with four-place accuracy) in the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\,\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0,$$

determine the value of n.

Forming the table of differences we have

x	y	Δ	Δ^2	Δ^3	Δ^4
1·0	0·765198				
		−45576			
1·1	0·719622		−2913		
		−48489		355	
1·2	0·671133		−2558		19
		−51047		374	
1·3	0·620086	(−52139)	−2184	(381)	14
		−53231		388	
1·4	0·566855		−1796		9
		−55027		397	
1·5	0·511828		−1399		
		−56426			
1·6	0·455402				

Here $w = 0{\cdot}1, \quad a = 1{\cdot}3, \quad f_0 = f(a) = f(1{\cdot}3).$

Hence
$$0{\cdot}1 f'(1{\cdot}3) = -0{\cdot}052139 - \tfrac{1}{6}(0{\cdot}000381)$$
$$f'(1{\cdot}3) = -0{\cdot}52203$$
$${\cdot}01 f''(1{\cdot}3) = -0{\cdot}002184 - \tfrac{1}{12}(0{\cdot}000014)$$
$$f''(1{\cdot}3) = -0{\cdot}2185.$$

Substituting these in
$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0,$$
we have
$$-0{\cdot}2185 + \frac{1}{1{\cdot}3}(-0{\cdot}5220) + \left(1 - \frac{n^2}{(1{\cdot}3)^2}\right)(0{\cdot}6201) = 0$$
$$-0{\cdot}2185 - 0{\cdot}4016 + \left(1 \quad \frac{n^2}{(1{\cdot}3)^2}\right)(0{\cdot}6201) = 0$$
$$-0{\cdot}6201 + \left(1 - \frac{n^2}{(1{\cdot}3)^2}\right)(0{\cdot}6201) = 0$$
$$-1 + 1 - \frac{n^2}{(1{\cdot}3)^2} = 0$$
$$n = 0.$$

Hence the Bessel function in question is $J_0(x)$.

Example 5.—Show that the following observations are these of a periodic function, and find its period :—

x (years)	$f(x)$
1	0·198669
2	0·295520
3	0·389418
4	0·479425
5	0·564642
6	0·644217
7	0·717356
8	0·783327

Example 6. – Observations were taken of the cooling of a boiler, and the results are given in the following table: –

t (time in hours)	1·2	1·9	2·6	3·3	4·0
θ (temperature °F)	182·3	175·2	171·1	164·3	160·6

Determine as carefully as you can the value of $\frac{d\theta}{dt}$ when $t=2{\cdot}6$ hours.

MISCELLANEOUS EXAMPLES.

1. Find a polynomial of the third degree in x which assumes the values 37, 11, 13, 29 when x is equal to 2, 1, -1, -2 respectively. Also find the minimum value of the function.

2. A steam electric generator on three long trials, each with a different point of cut off on steady load, is found to use the following amounts of steam per hour for the following amounts of power: –

Lb. of steam per hour	4020	6650	10800
Indicated horse-power	210	480	706
Kilowatts produced	114	290	435

Find the indicated horse-power and the weight of steam used per hour when 330 kilowatts are being produced.

3. Using first differences only, find the value of $f(4817{\cdot}36)$ if

x	4816	4817	4818
$f(x)$	693974063	694046108	694118146

To what extent may the result be relied on?

4. Interpolate for $x=35{\cdot}2967$ from the following table and state the accuracy of the result: —

x	$f(x)$
35·28	162021·40139
35·29	162043·22198
35·30	162065·04777

5. If the sun's apparent longitude at G.M.N. on 1914 Aug. 1st was 128° 22′ 25″·8; on Aug. 3rd 130° 17′ 15″·2; on Aug. 5th 132° 12′ 7″·8; find the longitude at G.M.N. on Aug. 2nd.

6. At the following draughts in sea water a particular vessel has the following displacements: —

Draught in feet .	15	12	9	6
Displacement in tons	2098	1512	1018	536

What are the probable displacements when the draughts are 11 and 13 feet respectively?

7. The following table gives the expectation of life for males of the ages mentioned: —

Present age	5	10	15	25	35
Expectation of life in years	50·87	47·60	43·41	35·68	28·64

Find the expectations of 3 men whose ages are 18, 20, 30 years respectively.

8. The H^M table of the Institute of Actuaries gives the following values of l_x :—

x	l_x
10	100000
12	99113
14	98496
16	97942
18	97245
20	96223

Obtain the values of l_x when x has the values 11, 13, 15, 17, 19.

9. The sun's apparent right ascension at mean noon is given in the following table :—

	H.	M.	S.
1914, June 1	4	34	0·69
6	4	54	31·75
11	5	15	10·64
16	5	35	55·34
21	5	56	43·18
26	6	17	30·99

Determine the apparent right ascension for 1914 June 15d. 16h.

10. Let the number of recruits in the United States army whose height did not exceed x inches be denoted by y. Determine the number whose height exceeded 66 inches, but did not exceed $66\frac{1}{2}$ inches, if

x	y
62	618
63	1855
64	3802
65	6821
66	10296
67	14350
68	17981
69	21114
70	23189
71	24674

11. The sun's apparent declination ∂ at mean noon was

1914, May 1	+14°	54′	19″·6
3	+15°	30′	25″·0
5	+16°	5′	28″·4
7	+16°	39′	27″·6
9	+17°	12′	20″·3
11	+17°	44′	4″·3

Determine the hourly difference in ∂ for 1914 May 5 and 7.

12. In the following table s denotes the space described at the time t by a point in a mechanism. Tabulate the values of the velocity and the acceleration, and obtain their values when $t=0{\cdot}45$.

t (time in secs.)	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1
s (space in feet)	0	0·032	0·12	0·281	0·525	0·854	1·260	1·725	2·229	2·753	3·282	3·811

13. There is a piece of mechanism whose weight is 200 lbs. The following values of s in feet show the distance of its centre of gravity from some point in its straight path at the time t seconds from some era of reckoning. Find its acceleration at the time $t=2{\cdot}05$, and the force in pounds which is giving this acceleration to it.

t	2·00	2·02	2·04	2·06	2·08	2·10
s	0·3090	0·4931	0·6799	0·8701	1·0643	1·2631

14. The following values of p (lbs. per sq. foot) and θ°C are for saturated steam. Calculate with as much accuracy as the numbers will allow the value of $\frac{dp}{d\theta}$ for $\theta=115$°C.

θ	105	110	115	120	125
p	2524	2994	3534	4152	4854

15. From the following table of the secular variations of the magnetic declination determine the chief period :—

Year.	Declination.
1540·5	− 7°·463
1580·5	− 8°·500
1620·5	− 6°·250
1660·5	− 0°·500
1700·5	+ 8°·228
1740·5	+ 15°·755
1780·5	+ 20°·832
1820·5	+ 22°·380
1860·5	+ 19°·364
1900·5	+ 14°·833

CHAPTER III

THE CONSTRUCTION AND USE OF MATHEMATICAL TABLES

13. Interpolation by First Differences.

All mathematical tables are alike in that they give numerical values of the function for certain values of the argument, which are so chosen that intermediate values of the function may be derived by interpolation. The interval of the argument varies in the different mathematical tables, its choice in each case being determined by the rapidity of variation of the function. In most of the published elementary tables, which are intended for the use of students, the interval of the argument is so chosen that the second difference is smaller than one unit in the last place of decimals retained: on this account it is not necessary to consider second differences in the interpolation, and the function can be calculated for any value of the argument intermediate between two of the tabulated values by a simple proportion in the following manner:—

Consider a function $f(x)$ which is tabulated for the values x_0, x_0+w, x_0+2w, ... of the argument. Then, since the function increases uniformly, when digits beyond a certain place of decimals are neglected, we have

$$\frac{f(x_0+w')-f(x_0)}{f(x_0+w)-f(x_0)} = \frac{w'}{w}$$

where $0<w'<w$.

From this it follows that

$$f(x_0+w') = f(x_0) + \frac{w'}{w}\{f(x_0+w) - f(x_0)\},$$

so that, if the first difference $f(x_0+w)-f(x_0)$ is known, the value of the function for any value of x between x_0 and x_0+w may be found. This equation is known as the "Rule of Proportional Parts." Geometrically, what we do is to replace the curve through two consecutive points by the chord joining them.

If the first differences are only sensibly constant, the result obtained will be subject to a certain error. If $w' = nw$ where $0 < n < 1$, then the error may be put in the approximate form

$$\frac{n(n-1)}{2} w^2 f''(x + n'w)$$

where $0 < n' < 1$.

Example 1.—In the following table the entry is the cube root of the corresponding argument. Find the cube root of 1619·25.

Argument.	Entry.	Δ
1619	11·7421858	
		24171
1620	11·7446029	
		24161
1621	11·7470190	

If we desire accuracy up to the fifth decimal place, then we may regard the first differences as constant.

Here $x_0 = 1619 \quad w = 1 \quad w' = 0{\cdot}25.$

Hence $\sqrt[3]{1619{\cdot}25} = 11{\cdot}74219 + \frac{25}{100}(0{\cdot}00242)$
$= 11{\cdot}74280.$

Example 2. — If $x = 1000$ show that, when second differences are neglected, the error committed in computing by this method the value of $\log_{10}(1000 + w)$, where $0 < w < 1$, will have no effect on the sixth decimal place of the logarithm.

Here $\log(x + w) = \log x + w\{\log(x+1) - \log x\}$

and the error committed is

$$E = \frac{w(1-w)}{2\,!} f''(x + w')$$

where $0 < w' < 1$ and $f(x) = \log_{10} x$.

Now $$\left| f''(x + w') \right| = \left| \frac{d^2}{dx^2} \log_{10}(x + w') \right|$$

$$= \left| -\frac{m}{(x + w')^2} \right|$$

where m = modulus of common logarithms = 0·43429.

But the maximum value of $w(1-w)$ is $\frac{1}{4}$ since

$$w(1-w) = \tfrac{1}{4} - (\tfrac{1}{2} - w)^2 \leqq \tfrac{1}{4}.$$

Hence the error committed is less than

$$\tfrac{1}{8} \cdot \frac{0{\cdot}43429}{(1000 + w')^2},$$

i.e. $$E < \frac{1}{16 \cdot 10^6}$$
$$< 0{\cdot}0000000625,$$

which shows that the error committed does not affect the sixth decimal place in the logarithm of the number $(1000+w)$.

Example 3.—The differences between the six-figure logarithms, to base 10, of the successive integers between 5000 and 5100 are sensibly constant. Prove this and determine the value of the constant difference.

14. Case of Irregular Differences.

When the second differences cannot be neglected, however, this method cannot be applied. This happens frequently when the rate of change of the function is large compared with that of the argument, as is the case with the logarithmic sines and tangents of small angles. We must then use the formulae of Chapter II., including differences to the order required.

Example 1.—To obtain $\log \sin 0° \, 16' \, 24''\cdot 5$.

From Schrön's tables we have

	Δ	Δ^2	Δ^3
$L \sin 0° \, 16' \, 10'' = 7\cdot 6723450$			
	44543		
$L \sin 0° \, 16' \, 20'' = 7\cdot 6767993$		-452	
	44091		9
$L \sin 0° \, 16' \, 30'' = 7\cdot 6812084$		-443	
	43648		
$L \sin 0° \, 16' \, 40'' = 7\cdot 6855732$			

Using Gauss' formula

$$f(a+nw) = f_0 + n\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!}\,\delta^2 f_0 + \frac{(n+1)\,n\,(n-1)}{3!}\,\delta^3 f_{\frac{1}{2}}$$

we obtain

$$\begin{aligned} L \sin 0° \, 16' \, 24''\cdot 5 &= 7\cdot 6767993 - 1 \\ &\qquad\qquad\quad 19841 \\ &\qquad\qquad\qquad\; 56 \\ &= 7\cdot 6787889 \end{aligned}$$

This is the answer correct to 7 places, the value correct to 10 places being found from Vega to be $7\cdot 6787889383$.

The value obtained by the "Rule of Proportional Parts" is $7\cdot 6787834$. This result is correct only to 4 places of decimals, and shows the necessity of interpolating by some such method as the above the value of the logarithmic sine of a small angle. This may be shown to be also true in the case of the logarithmic tangents of small angles.

Example 2.—Compute the value of $\log \tan 0° \, 15' \, 35''$.

15. Construction of New Tables.

In spite of the large number of different mathematical tables that have been constructed to assist the computer, further research in the various branches of mathematics and the kindred sciences shows that these are not sufficient for all purposes. A research student often requires new tables for the question which he is discussing, and these he must construct for himself.

The first question to be settled is the degree of accuracy to which the tables are to be computed, *i.e.*, the number of decimal places to be retained. This varies in different cases; but it is always desirable to extend the original calculations beyond the number of places which are ultimately to be retained, in order to avoid the cumulative effect of the neglected digits.

The general plan of the calculation is to compute first with great accuracy the values of the function for a series of widely-spaced values of the argument. For this purpose infinite series are generally employed. The values of the function for a more closely-spaced series of values of the argument are then derived from these by means of the formulae of Chapter II.: this second process is called *Subtabulation*. We shall consider these processes in §§ 17, 18.

16. Tabulation of a Polynomial.

The simplest case to be considered is the tabulation of a polynomial.

If the degree of $f(x)$ is n, it will not be necessary to compute its value directly for more than $(n+1)$ values of the variable, as may be seen from the following examples.

Example 1.—Let $f(x)=x^3+3x^2+2x-1$.

When $x=0, 1, 2, 3$, we have $f(x)=-1, 5, 23, 59$. We thus obtain the following table:—

x	$f(x)$	Δ	Δ^2	Δ^3
0	−1			
		6		
1	5		12	
		18		6
2	23		18	
		36		
3	59			

Since $f(x)$ is of the third degree in x, its third differences will be constant; in this case they will be equal to 6. Extending the column Δ^3 by inserting 6's we can then extend the column Δ^2 by simple additions or subtractions. A repetition of the same process will enable us to extend the column Δ, and

thereafter to determine the values of $f(x)$ for all integral values of x. The resulting table is

x	$f(x)$	Δ	Δ^2	Δ^3
-3	-7		-12	
		6		6
-2	-1		-6	
		0		6
-1	-1		0	
		0		6
0	-1		6	
		6		6
1	5		12	
		18		6
2	23		18	
		36		6
3	59		24	
		60		6
4	119		30	
		90		6
5	209		36	

By this simple method we are enabled to determine the values of a polynomial $f(x)$ for all positive and negative integral values of x. We can then use any of the formulae of interpolation already established to obtain the values of $f(x)$ for fractional values of x. The method may be extended to functions which are not rational or integral, but in this case it will be necessary to make the interval between successive values of x so small that the differences of some definite order are constant throughout the range of values under consideration.

Example 2.—Form the table of values of $f(x) = x^4 + 5x^3 - 7x^2 + x - 4$ for the values 0, 1, 2, 3, 4 of the argument x, and then deduce the values of $f(x)$ corresponding to the values -1, -2, -3, -4 of x.

17. Calculation of the Fundamental Values for a Table of Logarithms.

We shall now illustrate the process of computing tables of functions other than polynomials by describing the calculation of a table of logarithms.

The first stage in this consists in determining the logarithms of all the prime numbers between certain limits. It is evident that when the logarithms of the primes are known, the logarithms of all composite numbers can be derived from them by mere use of the formula

$$\log ab = \log a + \log b.$$

In order to obtain the logarithms of the primes we take two equations whose roots are integers and which differ only in their constant term. Such a pair are

$$x^3 - 3x + 2 = 0$$
$$x^3 - 3x - 2 = 0.$$

Now
$$x^3 - 3x + 2 = (x-1)^2 (x+2)$$
$$x^3 - 3x - 2 = (x+1)^2 (x-2).$$

Hence

$$\log(x-1) + \tfrac{1}{2}\log(x+2) - \log(x+1) - \tfrac{1}{2}\log(x-2)$$
$$= \tfrac{1}{2}\log\frac{x^3 - 3x + 2}{x^3 - 3x - 2}$$
$$= \tfrac{1}{2}\log\frac{1 + \dfrac{2}{x^3 - 3x}}{1 - \dfrac{2}{x^3 - 3x}}$$

which, by use of the ordinary logarithmic series, may be written, if the logarithms are all to the base e,

$$= \frac{2}{x^3 - 3x} + \tfrac{1}{3}\left\{\frac{2}{x^3 - 3x}\right\}^3 + \tfrac{1}{5}\left\{\frac{2}{x^3 - 3x}\right\}^5 + \dots$$

This is called *Borda's Formula.*

Putting $x = 5, 6, 7, 8$, we obtain the four equations

$$\log 2 - \tfrac{3}{2}\log 3 + \tfrac{1}{2}\log 7 = 0{\cdot}0181838220854376\dots$$
$$\tfrac{1}{2}\log 2 + \log 5 - \log 7 = 0{\cdot}0101013536587597\dots$$
$$-2\log 2 + 2\log 3 - \tfrac{1}{2}\log 5 = 0{\cdot}0062112599992784\dots$$
$$-\tfrac{5}{2}\log 3 + \tfrac{1}{2}\log 5 + \log 7 = 0{\cdot}0040983836020895\dots$$

Solving these four equations we obtain the logarithms (to the base e) of the prime numbers 2, 3, 5, 7, viz.:—

$$\log 2 = 0{\cdot}6931471805599453\dots$$
$$\log 3 = 1{\cdot}0986122886681096\dots$$
$$\log 5 = 1{\cdot}6094379124341003\dots$$
$$\log 7 = 1{\cdot}9459101490553133\dots$$

Further substitution for x in Borda's formula will give immediately the logarithms of the other prime numbers. For example, on putting $x = 9$ the left-hand side becomes

$$\log 8 + \tfrac{1}{2}\log 11 - \log 10 - \tfrac{1}{2}\log 7.$$

Of these log 7 has been found;

$$\log 8 \;= 3\log 2 = 2{\cdot}0794415416798359\ldots;$$

and $$\log 10 = \log 2 + \log 5 = 2{\cdot}3025850929940456\ldots.$$

Hence we obtain $\log 11 = 2{\cdot}3978952727983705\ldots.$

It is to be noted that these logarithms are to the base e. But as we have now found $\log_e 10$, we can at once derive its reciprocal $\log_{10} e = 0{\cdot}4342944819032518\ldots$, and all the above logarithms can now be converted to the base 10 by multiplying them by this "modulus."

We thus obtain the table

$$\begin{aligned}
\log_{10} 2 &= 0{\cdot}3010299956639812\\
\log_{10} 3 &= 0{\cdot}4771212547196624\\
\log_{10} 5 &= 0{\cdot}6989700043360188\\
\log_{10} 7 &= 0{\cdot}8450980400142568\\
\log_{10} 11 &= 1{\cdot}0413926851582250\\
&\text{etc.}
\end{aligned}$$

For the calculation of the logarithms of the larger primes *Haros' Formula* will be found more useful. In this case consider the equations

$$x^4 - 25x^2 = 0$$
$$x^4 - 25x^2 + 144 = 0.$$

From these we deduce

$$\log\frac{(x+5)(x-5)x^2}{(x+3)(x-3)(x+4)(x-4)} = \log\frac{x^4-25x^2}{x^4-25x^2+144} = \log\frac{(x^4-25x^2+72)-72}{(x^4-25x^2+72)+72}$$

whence, if the logarithms are to the base 10, we have

$$\log(x+5) = \{\log(x+3)+\log(x-3)+\log(x+4)+\log(x-4)\} - \{\log(x-5)+2\log x\}$$
$$-2m\left\{\frac{72}{x^4-25x^2+72} + \tfrac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + \ldots\right\}$$

where $m = \log_{10} e$.

Example 1.—To obtain the logarithm of the prime number 43 to the base 10.

Let $$x+5=43.$$
Then $$x=38$$

$$\log 43 = \log 41 + \log 35 + \log 42 + \log 34 - \log 33 - 2\log 38$$
$$-0{\cdot}8685889638065036\,\{0{\cdot}0000351372402040 + 0{\cdot}0000000000000145\}.$$

But the logarithms of 41, 35, 42, 34, 33, 38 are assumed to have been previously calculated. Hence

$$\begin{aligned}
\log 43 = {} & 1{\cdot}6127838567197355 \quad - 1{\cdot}5185139398778875\\
& 1{\cdot}5440680443502756 \quad\;\; 3{\cdot}1595671932336203\\
& 1{\cdot}6232492903979005 \quad\;\; 0{\cdot}0000305198190724\\
& 1{\cdot}5314789170422551\\
= {} & 1{\cdot}6334684555795865
\end{aligned}$$

Example 2.—Show that $\log_{10} 131 = 2{\cdot}1172712956557643.$

18. Amplification of the Table by Subtabulation.

Having now obtained the logarithms of the prime numbers within a certain range, we derive from them, by mere addition, the logarithms of the composite numbers which lie in this range, and thus obtain the logarithms of all the integers within the range. We shall now show how, by subtabulation, the extent of this table may be increased manyfold. It will be supposed that 7-place accuracy is required.

For example, suppose the logarithms of the integers 31, 32, 33, 34, 35, 36 have been obtained directly; we shall show how the logarithms of all the integers between 330 and 340 can be derived from them.

Neglecting the characteristics of the logarithms, which can always be written down by inspection, we form the following table:—

x	$\log x$	Δ	Δ^2	Δ^3	Δ^4
31	·49136169				
		1378829			
32	·50514998		−42433		
		1336396		2535	
33	·51851394		−39898		−223
		1296498		2312	(−208)
34	·53147892		−37586		−192
		1258912		2120	
35	·54406804		−35466		
		1223446			
36	·55630250				

Putting $a = 33$, $n = \frac{1}{10}$, $w = 1$ in Gauss' formula

$$f(a+nw) = f_0 + n\,\delta f_{\frac{1}{2}} - \frac{n(1-n)}{2\,!}\,\delta^2 f_0 - \frac{n(1-n^2)}{3\,!}\,\delta^3 f_{\frac{1}{2}} + \frac{n(1-n^2)(2-n)}{4\,!}\,\mu\,\delta^4 f_{\frac{1}{2}}$$

we obtain

$$\begin{aligned} f(33{\cdot}1) &= f_0 + 0{\cdot}1\,\delta f_{\frac{1}{2}} - 0{\cdot}045\,\delta^2 f_0 - 0{\cdot}0165\,\delta^3 f_{\frac{1}{2}} + 0{\cdot}0081\,\mu\,\delta^4 f_{\frac{1}{2}} \\ &= {\cdot}51851394 - 38 \\ &\phantom{= {\cdot}51}129650 \quad\ 1 \\ &\phantom{= {\cdot}518}1795 \\ &= {\cdot}51982800. \end{aligned}$$

Thus to 7 decimal places $\log 331 = 2{\cdot}5198280$. But it is not necessary to calculate the values for all the numbers between 330 and 340 in this way. It is, in fact, simpler to calculate $\log 332$ from the formula

$$f(a+\tfrac{2}{10}w) = f(a+\tfrac{1}{10}w) + \{f(a+\tfrac{1}{10}w) - f(a)\} + \delta_2$$

$$\text{where}\quad \delta_2 = f(a+\tfrac{2}{10}w) - 2f(a+\tfrac{1}{10}w) + f(a)$$

$$= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}001\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}001\,\mu\,\delta^4 f_{\frac{1}{2}}$$

and then to calculate $\log 333$ from the formula

$$f(a+\tfrac{3}{10}w) = f(a+\tfrac{2}{10}w) + \{f(a+\tfrac{2}{10}w) - f(a+\tfrac{1}{10}w)\} + \delta_3$$

$$\text{where}\quad \delta_3 = f(a+\tfrac{3}{10}w) - 2f(a+\tfrac{2}{10}w) + f(a+\tfrac{1}{10}w)$$

$$= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}002\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0017\,\mu\,\delta^4 f_{\frac{1}{2}}.$$

The complete table of these quantities δ is as follows:—

$$\begin{aligned}
\delta_2 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}001\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0010\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_3 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}002\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0017\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_4 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}003\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0019\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_5 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}004\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0021\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_6 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}005\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0020\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_7 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}006\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0021\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_8 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}007\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0018\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_9 &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}008\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0017\,\mu\,\delta^4 f_{\frac{1}{2}}\\
\delta_{10} &= 0{\cdot}01\,\delta^2 f_0 + 0{\cdot}009\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}0012\,\mu\,\delta^4 f_{\frac{1}{2}}
\end{aligned}$$

The numerical work is arranged as in the following table, in which the logarithms required are found in the last column.

	− 39898·0 +	2312·0 −	208·0		129649·8	2·51851394
$\frac{1}{10}$	+ 1795·4 −	38·1 −	1·7 =	− 1755·6	131405·4	2·519827994
$\frac{2}{10}$	− 399·0 +	2·3 +	0·2 =	− 396·5	131008·9	2·521138083
$\frac{3}{10}$	− 399·0 +	4·6 +	0·4 =	− 394·0	130614·9	2·522444232
$\frac{4}{10}$	− 399·0 +	6·9 +	0·4 =	− 391·7	130223·2	2·523746464
$\frac{5}{10}$	− 399·0 +	9·2 +	0·4 =	− 389·4	129833·8	2·525044802
$\frac{6}{10}$	− 399·0 +	11·6 +	0·4 =	− 387·0	129446·8	2·526339270
$\frac{7}{10}$	− 399·0 +	13·9 +	0·4 =	− 384·7	129062·1	2·527629891
$\frac{8}{10}$	− 399·0 +	16·2 +	0·4 =	− 382·4	128679·7	2·528916688
$\frac{9}{10}$	− 399·0 +	18·5 +	0·4 =	− 380·1	128299·6	2·530199684
$\frac{10}{10}$	− 399·0 +	20·8 +	0·2 =	− 378·0	127921·6	2·531478900

The method followed in constructing this table will best be shown by considering two examples.

In the line $\frac{1}{10}$

$+1795{\cdot}4$ is the numerical value obtained for $0{\cdot}045\ \delta^2 f_0$

$-\ \ 38{\cdot}1$,, ,, ,, $-0{\cdot}0165\ \delta^3 f_{\frac{1}{2}}$

$-\ \ 1{\cdot}7$,, ,, ,, $0{\cdot}008\ \mu\,\delta^4 f_{\frac{1}{2}}$

$1755{\cdot}6$ is the sum of the three preceding numbers.

$129649{\cdot}8$ is the numerical value obtained for $0{\cdot}01\ \delta f_{\frac{1}{2}}$.

$131405{\cdot}4$ is the sum of the two precediog numbers, and therefore represents

$$0{\cdot}01\,\delta f_{\frac{1}{2}} + 0{\cdot}045\,\delta^2 f_0 - 0{\cdot}0165\,\delta^3 f_{\frac{1}{2}} + 0{\cdot}008\,\mu\,\delta^4 f_{\frac{1}{2}}.$$

$2{\cdot}51851394$ is the numerical value of f_0

∴ $2{\cdot}519827994$, which is obtained as the sum of the two preceding numbers, represents

$$f_0 + 0{\cdot}01\,\delta f_{\frac{1}{2}} + 0{\cdot}045\,\delta^2 f_0 - 0{\cdot}0165\,\delta^3 f_{\frac{1}{2}} + 0{\cdot}008\,\mu\,\delta^4 f_{\frac{1}{2}}.$$

In the line $\frac{2}{10}$

$-399{\cdot}0$ is the numerical value obtained for $0{\cdot}01\ \delta^2 f_0$

$+\ \ 2{\cdot}3$,, ,, ,, $0{\cdot}001\ \delta^3 f_{\frac{1}{2}}$

$+\ \ 0{\cdot}2$,, ,, ,, $-0{\cdot}001\ \mu\,\delta^4 f_{\frac{1}{2}}$

$-396{\cdot}5$, which is obtained by summing the three preceding numbers, represents

$$0{\cdot}01\,\delta^2 f_0 + 0{\cdot}001\,\delta^3 f_{\frac{1}{2}} - 0{\cdot}001\,\mu\,\delta^4 f_{\frac{1}{2}} \text{ or } \delta_2.$$

$131405{\cdot}4$, which has previously been obtained, represents

$$f(a + \tfrac{1}{10}w) - f(a).$$

$131008{\cdot}9$, which is obtained by summing the numbers immediately above it and to the left of it, represents

$$f(a + \tfrac{1}{10}w) - f(a) + \delta_2.$$

Now $2{\cdot}519827794$, as we have already seen, represents $f(a + \frac{1}{10}w)$.

∴ $2{\cdot}521138083$, which is obtained by adding the numbers immediately above it and to the left of it, represents

$$f(a + \tfrac{2}{10}w).$$

The results in the last column are the logarithms of the numbers 330, 331, ... 340. They cannot, of course, be relied on up to the last figure, but will be found correct up to seven decimal places. This accuracy may be obtained by taking one decimal more in the first difference than in the tabular function, one decimal more in the second difference than in the first, and so on, when these can be obtained. A check on the accuracy is obtained by carrying on the process until log 340 is computed. Comparison of this result with the value for log 34 will show whether the computation has been accurately performed.

We may remark that it is an improvement on the foregoing process if the differences required for the subtabulation are computed independently from the expressions in series, instead of being derived by forming a table of differences.

Example 1.—Given
$$\log 24 = 1{\cdot}3802112$$
$$\log 25 = 1{\cdot}3979400$$
$$\log 26 = 1{\cdot}4149733$$
$$\log 27 = 1{\cdot}4313638$$
$$\log 28 = 1{\cdot}4471580$$
$$\log 29 = 1{\cdot}4623980$$
obtain the logarithms of the numbers 261, 262, ... 269.

19. Radix Method of Calculating Logarithms.

When the logarithm of a number is required to a large number of decimal places, its determination either by means of an infinite series or by interpolation becomes very laborious and liable to error. A much simpler and more useful method in such a case is that known as the *Radix Method.*

Let N be the given number whose logarithm is required to n places of decimals. Then since

$$N = 10^x \,.\, a \,.\, (1 + N_0)$$

where N_0 is a decimal, x an integer, and $a = 0, 1, 2, \ldots 9$,

$$\log_{10} N = x + \log_{10} a + \log_{10} (1 + N_0).$$

Thus, to determine $\log_{10} N$, it will be sufficient to compute $\log_{10} a$ and $\log_{10} (1 + N_0)$. The former has been obtained above. We shall now show how to evaluate the latter.

Let r_1 denote the decimal consisting of the first m figures in N_0, and let N_1 be the difference of N_0 and r_1. Then

$$(1 + N_0) - (1 + r_1) = N_1.$$

Now divide N_1 by $(1 + r_1)$ until there are m figures in the quotient r_2. If we call the remainder N_2, then

$$N_1 - (1 + r_1)\, r_2 = N_2.$$

Similarly, if the remainder, after dividing N_2 by $(1 + r_1)(1 + r_2)$ until there are m figures in the quotient r_3, is denoted by N_3, we have

$$N_2 - (1 + r_1)(1 + r_2)\, r_3 = N_3,$$

and in general,

$$N_{p-1} - (1 + r_1)(1 + r_2) \ldots (1 + r_{p-1})\, r_p = N_p.$$

These equations may be written in the form

$$\begin{aligned}
(1+N_0) \quad - N_1 &= (1+r_1) \\
N_1 \quad - N_2 &= (1+r_1)(1+r_2) - (1+r_1) \\
N_2 \quad - N_3 &= (1+r_1)(1+r_2)(1+r_3) - (1+r_1)(1+r_2) \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
N_{p-2} - N_{p-1} &= (1+r_1)(1+r_2)\ldots(1+r_{p-1}) \\
&\qquad - (1+r_1)(1+r_2)\ldots(1+r_{p-2}) \\
N_{p-1} - N_p &= (1+r_1)(1+r_2)\ldots(1+r_p) \\
&\qquad - (1+r_1)(1+r_2)\ldots(1+r_{p-1}).
\end{aligned}$$

Adding, we obtain

$$(1+N_0) \quad - N_p = (1+r_1)(1+r_2)\ldots(1+r_p)$$

whence

$$1+N_0 = (1+r_1)(1+r_2)\ldots(1+r_p)+N_p.$$

If, therefore, the remainder N_p has no significant figures in its first n decimal places,

$$\log(1+N_0) = \log(1+r_1)+\log(1+r_2)+ \ldots +\log(1+r_p).$$

The quantities $(1+r_1)$, $(1+r_2)$, ... are called the *radices.* As they will be of the form $\left(1+\dfrac{k}{10^l}\right)$ where k and l are integers, and as k is always small compared with 10^l we may compute $\log(1+r_p)$ by means of the logarithmic expansion

$$\log(1+r_p) = r_p - \tfrac{1}{2}r_p^2 + \tfrac{1}{3}r_p^3 - \tfrac{1}{4}r_p^4 + \ldots$$

The values of the logarithms of these radices when $k=0, 1, 2, \ldots 999$, and $l=3, 6, 9, 12, 15, 18, 21, 24$, as well as of the numbers 1, 2, ... 9, have been computed to twenty-four places of decimals.*

The inverse process of determining the antilogarithm corresponding to a given logarithm is equally simple. When the logarithm is not in the table, we have merely to take out the next lower and subtract it from the given one. Then take out the next lower and subtract it from this remainder; and so on until the remainder consists of zeros. The product of the radices corresponding to the logarithms taken out is the number required.

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* Gray, *Tables for the formation of Logarithms and Antilogarithms to twenty-four or any less number of places.* London, C. & E. Layton (1900).

We shall illustrate these two methods in the following examples :—

Example 1.—To determine $\log_{10} 43$ to 24 places of decimals.

Here $x=1$, $a=3$.

Taking $m=3$, as in Gray's tables, we have

$$1+N_0=1\cdot 433\,333\,333\,333\,333\,333\,333\,333$$
$$1+r_1=1\cdot 433$$
$$N_1=0\cdot 000\,333\,333\,333\,333\,333\,333\,333$$
$$r_2=0\cdot 000\,232$$
$$N_2=0\cdot 000\,000\,877\,333\,333\,333\,333\,333$$
$$(1+r_1)\,r_2=0\cdot 000\,332\,456 \quad i.e.,\ 0\cdot 000\,333\,333-0\cdot 000\,000\,877$$
$$(1+r_1)(1+r_2)=(1+r_1)+(1+r_1)\,r_2$$
$$=1\cdot 433\,332\,456.$$

The above will be sufficient to indicate the general theory. The process can, however, be simplified by arranging the work in the following manner :—

```
       1·433 333 333 333 333 333 333 333
       1·433
Diff. =0·000 333 333 333 333 333 333 333
        1·433 ) 0·000 333 333 333 333 333 333 333 ( 0·000 232
                      286 6
                      46 73
                      42 99
                       3 743
                       2 866
                         877 = Remainder.
       1·433 333 333
       0·000 000 877
Diff. =1·433 332 456 = New Divisor.
     1·433 332 456 ) 0·000 000 877 333 333 333 333 333 ( 0·(000)²612 *
                          859 999 473 6
                           17 333 859 73
                           14 333 324 56
                            3 000 535 173
                            2 866 664 912
                              133 870 261 = Remainder.
       1·433 333 333 333 333 333
       0·000 000 000 133 870 261
Diff. =1·433 333 333 199 463 072 = New Divisor.
```

Should the original dividend not be known beyond the 24th place of decimals, we should now have to proceed by a method of contracted division. In the present case we know the figures after the 24th decimal, and may therefore continue as above. But we shall adopt the contracted method.

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* $0\cdot(000)^2 612$ is written in place of $0\cdot 000\,000\,612$
$0\cdot(000)^3 093$,, ,, $0\cdot 000\,000\,000\,093$ and so on.

The new divisor contains 19 figures ; the new dividend

$$\cdot(000)^3\,133\,870\,261\,333\,333$$

contains only 15 significant figures after the point. We must therefore cut off 4 figures from the divisor. Thus we obtain

$$\begin{array}{r}
1\cdot433\,333\,333\,199_{\text{,}}4_{\text{,}}6\,)\,0\cdot(000)^3\,133\,870\,261\,333\,333\,(\,0\cdot(000)^3\,093\,*\\
128\,999\,999\,987\,951\\
\hline
4\,870\,261\,345\,382\\
4\,299\,999\,999\,598\\
\hline
570\,261\,345\,784 = \text{Remainder.}
\end{array}$$

$$\begin{array}{r}
1\cdot433\,333\,333\,333\,333\,333\,333\,333\\
0\cdot000\,000\,000\,000\,570\,261\,345\,784\\
\hline
\text{Diff.} = 1\cdot433\,333\,333\,332\,763\,071\,987\,549
\end{array} = \text{New Divisor.}$$

As the new dividend $0\cdot(000)^4\,570\,261\,345\,784$ contains only 12 significant figures and the divisor 25 we must cut off 13 figures from the latter. It then becomes $1\cdot433\,333\,333\,33$. As these are the figures in the original dividend there will be no further new divisors. This will usually happen when the fourth division is reached. We may therefore determine the remaining figures by continued contracted division.

$$\begin{array}{r}
1\cdot433\,333\,333\,33\,)\,0\cdot(000)^4\,570\,261\,345\,784\,(\,0\cdot(000)^4\,397\,856\,752\,873\\
430\,000\,000\,000\\
\hline
140\,261\,345\,784\\
129\,000\,000\,000\\
\hline
11\,261\,345\,784\\
10\,033\,333\,333\\
\hline
1\,228\,012\,451\\
1\,146\,666\,666\\
\hline
81\,345\,785\\
71\,666\,667\\
\hline
9\,679\,118\\
8\,600\,000\\
\hline
1\,079\,118\\
1\,003\,333\\
\hline
75\,785\\
71\,667\\
\hline
4\,118\\
2\,867\\
\hline
1\,251\\
1\,146\\
\hline
105\\
100\\
\hline
5\\
4\\
\hline
\end{array}$$

The remaining radices are thus

$$1\cdot(000)^4\,397$$
$$1\cdot(000)^5\,856$$
$$1\cdot(000)^6\,752$$
$$1\cdot(000)^7\,873$$

Using Gray's tables of the radices we have

$\log 3$	=	$0\cdot 477\,121\,254\,719\,662\,437\,295\,028$
$\log 1\cdot 433$	=	$0\cdot 156\,246\,190\,397\,344\,475\,994\,693$
$\log 1\cdot 000232$	=	$100\,744\,633\,875\,845\,680\,796$
$\log 1\cdot(000)^2\,612$	=	$265\,788\,141\,593\,627\,087$
$\log 1\cdot(000)^3\,093$	=	$40\,389\,386\,815\,124$
$\log 1\cdot(000)^4\,397$	=	$172\,414\,909\,316$
$\log 1\cdot(000)^5\,856$	=	$371\,756\,077$
$\log 1\cdot(000)^6\,752$	=	$326\,589$
$\log 1\cdot(000)^7\,873$	=	379
Sum	=	$0\cdot 633\,468\,455\,579\,586\,526\,405\,089$

Hence since $x=1$ the required logarithm is

$$\log 43 = 1\cdot 633\,468\,455\,579\,586\,526\,405\,089.$$

Example 2.—To determine the antilogarithm of

$$1\cdot 120\,619\,882\,724\,133\,396\,646\,450.$$

As the integral part only affects the characteristic we shall neglect it for the present.

Using Gray's tables we then obtain

		$0\cdot 120\,619\,882\,724\,133\,396\,646\,450$
$\log\ 1\cdot 320$	=	$0\cdot 120\,573\,931\,205\,849\,868\,472\,706$
Diff.	=	$45\,951\,518\,283\,528\,173\,744$
$\log\ 1\cdot 000105$	=	$45\,598\,526\,719\,080\,137\,349$
Diff.	=	$352\,991\,564\,448\,036\,395$
$\log\ 1\cdot(000)^2\,812$	=	$352\,646\,976\,130\,787\,551$
Diff.	=	$344\,588\,317\,248\,844$
$\log\ 1\cdot(000)^3\,793$	=	$344\,395\,524\,012\,726$
Diff.	=	$192\,793\,236\,118$
$\log\ 1\cdot(000)^4\,443$	=	$192\,392\,455\,483$
Diff.	=	$400\,780\,635$
$\log\ 1\cdot(000)^5\,922$	=	$400\,419\,512$
Diff.	=	$361\,123$
$\log\ 1\cdot(000)^6\,831$	=	$360\,899$
Diff.	=	224
$\log\ 1\cdot(000)^7\,516$	=	224
Diff.	=	0

Multiplying these radices together we have

$$1\cdot(000)^4\,443\times 1\cdot(000)^5\,922\times 1\cdot(000)^6\,831\times 1\cdot(000)^7\,516 = 1\cdot(000)^4\,443\,922\,831\,516.$$

This product of the last four factors can always be written down by inspection. The other multiplications may be arranged as follows :—

$$\begin{array}{r}
1\cdot000\,000\,000\,000\,443\,922\,831\,516 \\
1\cdot000\,000\,000\,793 \\
\hline
1\cdot000\,000\,000\,000\,443\,922\,831\,516 \\
700\,000\,000\,000\,310 \\
90\,000\,000\,000\,040 \\
3\,000\,000\,000\,000 \\
\hline
1\cdot000\,000\,000\,793\,443\,922\,831\,866 \\
1\cdot000\,000\,812 \\
\hline
1\cdot000\,000\,000\,793\,443\,922\,831\,866 \\
800\,000\,000\,634\,755\,138 \\
10\,000\,000\,007\,934\,439 \\
2\,000\,000\,001\,586\,888 \\
\hline
1\cdot000\,000\,812\,793\,444\,567\,108\,331 \\
1\cdot000\,105 \\
\hline
1\cdot000\,000\,812\,793\,444\,567\,108\,331 \\
100\,000\,081\,279\,344\,456\,711 \\
5\,000\,004\,063\,967\,222\,836 \\
\hline
1\cdot000\,105\,812\,878\,787\,878\,787\,878 \\
1\cdot320 \\
\hline
1\cdot000\,105\,812\,878\,787\,878\,787\,878 \\
300\,031\,743\,863\,636\,363\,636\,363 \\
20\,002\,116\,257\,575\,757\,575\,757 \\
\hline
1\cdot320\,139\,672\,999\,999\,999\,999\,998 \\
\hline
\end{array}$$

Since the characteristic in the given logarithm is 1, there will be 2 figures before the point in the corresponding antilogarithm. Thus the required result is

$$13\cdot201\,396\,729\,999\,999\,999\,999\,98.$$

Example 3.—Given $\pi=3\cdot141\,592\,653\,589\,793\,238\,462\,643$ show that

$$\log\pi=0\cdot497\,149\,872\,694\,133\,854\,351\,268.$$

Example 4.—Show that the antilogarithm of

$0\cdot434\,294\,481\,903\,251\,827\,651\,129$ is $2\cdot718\,281\,828\,459\,045\,235\,360\,290$.

20. Inverse Interpolation.

Suppose the values of a function have been tabulated, and that it is required to find the value of the argument which corresponds to some given value of the function, intermediate between two of the tabulated values. The process by which we obtain this value, which may be called the "antifunction," is known as *Inverse Interpolation.*

Let $f(a+nw)$ be the given value of the function. Then

$$f(a+nw) = f_0 + n\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!}\,\delta^2 f_1 + \dots$$
$$+ \frac{n(n-1)\dots(n-r+1)}{r!}\,\delta^r f_{\frac{r}{2}} + \dots$$

In this equation $f(a+nw)$ is given, and as the values of $f_0, f_1, \dots$ have been tabulated, the values of $\delta f_{\frac{1}{2}}$, $\delta^2 f_1$, ... are also known. Hence the only unknown quantity in the equation is n, and this is the number required for the determination of the anti-function $(a+nw)$.

To obtain it we must solve this equation. It has been pointed out already that the differences of the function under consideration usually become sensibly constant at some definite order, say r. The higher differences will therefore be zero, and this equation then becomes one of degree r in n. It can, therefore, be solved by a method of successive approximation, as follows:—

If we write the above equation in the form

$$n = \frac{f - f_0}{\delta f_{\frac{1}{2}} + \dfrac{n-1}{2!}\,\delta^2 f_1 + \dots}$$

and neglect differences of the second and higher orders, we get as a first approximation the value n_1 where

$$n_1 = \frac{f-f}{\delta f_{\frac{1}{2}}}\,.$$

Let us next take second differences into account and put n_1 for n in the denominator of the expression on the right-hand side. Then the next approximation n_2 is given by

$$n_2 = \frac{f-f_0}{\delta f_{\frac{1}{2}} + \frac{1}{2}(n_1-1)\,\delta^2 f_1}\,.$$

When third differences are taken into account, n_2 is substituted in the denominator for n_1, giving

$$n_3 = \frac{f-f_0}{\delta f_{\frac{1}{2}} + \frac{1}{2}(n_2-1)\,\delta^2 f_1 + \frac{1}{6}(n_2-1)(n_2-2)\,\delta^3 f_{\frac{3}{2}}}\,.$$

Proceeding in this way we obtain closer and closer approximations to the value of n. The method, though rather laborious, has the advantage that any error introduced in the computation of any

one approximation will not cause the final result to be wrong, being itself rectified in the higher approximations. Thus the sole effect of the error introduced is merely to make the approximation less rapid, and therefore to increase the amount of labour involved in the accurate determination of n. On this account the process is a safe one to employ.

Other formulae for determining the antifunction may be obtained by reverting any of the formulae of interpolation. We shall exemplify this by reverting Stirling's formula

$$f(a+nw)=f_0+n\,\mu\,\delta f_0+\frac{n^2}{2\,!}\,\delta^2 f_0+\frac{n(n^2-1)}{3\,!}\,\mu\,\delta^3 f_0$$
$$+\frac{n^2(n^2-1)}{4\,!}\,\delta^4 f_0+\frac{n(n^2-1)(n^2-4)}{5\,!}\,\mu\,\delta^5 f_0+\dots$$

Neglecting differences of higher order than the fifth, and arranging in powers of n, we have

$$f(a+nw)=f_0+(\mu\,\delta f_0-\tfrac{1}{6}\,\mu\,\delta^3 f_0+\tfrac{1}{30}\,\mu\,\delta^5 f_0)\,n+(\tfrac{1}{2}\,\delta^2 f_0-\tfrac{1}{24}\,\delta^4 f_0)\,n^2$$
$$+(\tfrac{1}{6}\,\mu\,\delta^3 f_0-\tfrac{1}{24}\,\mu\,\delta^5 f_0)\,n^3+\tfrac{1}{24}\,\delta^4 f_0\,n^4+\tfrac{1}{120}\,\mu\,\delta^5 f_0\,n^5$$

which may be written

$$F=An+Bn^2+Cn^3+Dn^4+En^5$$

where

$$\begin{aligned}
F&=f(a+nw)-f_0\\
A&=\mu\,\delta f_0-\tfrac{1}{6}\,\mu\,\delta^3 f_0+\tfrac{1}{30}\,\mu\,\delta^5 f_0\\
B&=\tfrac{1}{2}\,\delta^2 f_0-\tfrac{1}{24}\,\delta^4 f_0\\
C&=\tfrac{1}{6}\,\mu\,\delta^3 f_0-\tfrac{1}{24}\,\mu\,\delta^5 f_0\\
D&=\tfrac{1}{24}\,\delta^4 f_0\\
E&=\tfrac{1}{120}\,\mu\,\delta^5 f_0
\end{aligned}$$

Reverting this series we obtain successively

(i) $$n=\frac{F}{A}$$

(ii) $$n=\frac{F}{A}-\frac{B}{A}\left(\frac{F}{A}\right)^2$$

(iii) $$n=\frac{F}{A}-\frac{B}{A}\left\{\frac{F}{A}-\frac{B}{A}\left(\frac{F}{A}\right)^2\right\}^2-\frac{C}{A}\left(\frac{F}{A}\right)^3$$
$$=\frac{F}{A}-\frac{B}{A}\left(\frac{F}{A}\right)^2+\left\{2\left(\frac{B}{A}\right)^2-\frac{C}{A}\right\}\left\{\frac{F}{A}\right\}^3$$

$$\text{(iv)}\ n = \frac{F}{A} - \frac{B}{A}\left[\frac{F}{A} - \frac{B}{A}\left(\frac{F}{A}\right)^2 + \left\{2\left(\frac{B}{A}\right)^2 \quad \frac{C}{A}\right\}\left\{\frac{F}{A}\right\}^3\right]^2$$

$$- \frac{C}{A}\left\{\frac{F}{A} - \frac{B}{A}\left(\frac{F}{A}\right)^2\right\}^3 - \frac{D}{A}\left(\frac{F}{A}\right)^4$$

$$= \frac{F}{A} - \frac{B}{A}\left(\frac{F}{A}\right)^2 + \left\{2\left(\frac{B}{A}\right)^2 - \frac{C}{A}\right\}\left\{\frac{F}{A}\right\}^3$$

$$+ \left\{5\frac{BC}{A^2} - \frac{D}{A} - 5\left(\frac{B}{A}\right)^3\right\}\left\{\frac{F}{A}\right\}^4$$

$$\text{(v)}\ n = \frac{F}{A} - \frac{B}{A}\left(\frac{F}{A}\right)^2 + \left\{2\left(\frac{B}{A}\right)^2 - \frac{C}{A}\right\}\left\{\frac{F}{A}\right\}^3$$

$$+ \left\{5\frac{BC}{A^2} - \frac{D}{A} - 5\left(\frac{B}{A}\right)^3\right\}\left\{\frac{F}{A}\right\}^4$$

$$+ \left\{3\left(\frac{C}{A}\right)^2 + 6\frac{BD}{A^2} - \frac{E}{A} - 21\left(\frac{B}{A}\right)^2\frac{C}{A} + 14\left(\frac{B}{A}\right)^4\right\}\left\{\frac{F}{A}\right\}^5.$$

A more convenient form* may be obtained by putting $\frac{F}{A} = n_1$ and $\frac{n_1}{A} = r$ and rearranging the terms.

We then obtain the formula

$$n = n_1 + f_1 n_1 + f_2 n_1^2 + f_3 n_1^3 + f_4 n_1^4$$

where

$$f_1 = -Br + 2(Br)^2 - 5(Br)^3 + 14(Br)^4$$
$$f_2 = Cr\{-1 + 5Br - 21(Br)^2\}$$
$$f_3 = Dr(-1 + 6Br) + 3(Cr)^2$$
$$f_4 = -Er.$$

If the first derivate of the function $f(a + nw)$ is tabulated, the values of the quantities A, B, C, D, E may be obtained by forming the successive differences of $wf'(a + nw)$. As fourth differences of the derivate are of the same order as fifth differences of the function we need not extend the table beyond this order of differences.

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* Van Orstrand, *Phil. Mag.* (6) 15 (1908), p. 630.

Argument.	Entry.	Δ	Δ^2	Δ^3	Δ^4
$a-3w$	wf'_{-3}				
		$w\,\delta f'_{-\frac{5}{2}}$			
$a-2w$	wf'_{-2}		$w\,\delta^2 f'_{-2}$		
		$w\,\delta f'_{-\frac{3}{2}}$		$w\,\delta^3 f'_{-\frac{3}{2}}$	
$a-w$	wf'_{-1}		$w\,\delta^2 f'_{-1}$		$w\,\delta^4 f'_{-1}$
		$w\,\delta f'_{-\frac{1}{2}}$		$w\,\delta^3 f'_{-\frac{1}{2}}$	
a	wf'_0		$w\,\delta^2 f'_0$		$w\,\delta^4 f'_0$
		$w\,\delta f'_{\frac{1}{2}}$		$w\,\delta^3 f'_{\frac{1}{2}}$	
$a+w$	wf'_1		$w\,\delta^2 f'_1$		$w\,\delta^4 f'_1$
		$w\,\delta f'_{\frac{3}{2}}$		$w\,\delta^3 f'_{\frac{3}{2}}$	
$a+2w$	wf'_2		$w\,\delta^2 f'_2$		
		$w\,\delta f'_{\frac{5}{2}}$			
$a+3w$	wf'_3				

Differentiating the equation

$$f(a+nw)=f_0+An+Bn^2+Cn^3+Dn^4+En^5$$

we see that

$$A=wf'_0\,;\quad B=\tfrac{1}{2}\,w^2 f_0''\,;\quad C=\tfrac{1}{6}\,w^3 f_0'''\,;\quad D=\tfrac{1}{24}\,w^4 f_0^{(4)}$$

$$E=\tfrac{1}{120}\,w^5 f_0^{(5)}.$$

Now from Stirling's formula we have

$$wf'(a+nw)=wf'_0+n\,w\,\mu\,\delta f'_0+\frac{n^2}{2!}\,w\,\delta^2 f'_0+\frac{n(n^2-1)}{3!}\,w\,\mu\,\delta^3 f'_0+\frac{n^2(n^2-1)}{4!}\,w\,\delta^4 f'_0$$

$$\begin{aligned}
w^2 f''(a+nw)&=w\mu\delta f'_0+nw\delta^2 f'_0+\frac{3n^2-1}{6}\,w\,\mu\,\delta^3 f'_0+\frac{2n^3-n}{12}\,w\,\delta^4 f'_0\\
w^3 f'''(a+nw)&=w\,\delta^2 f'_0+n\,w\,\mu\,\delta^3 f'_0+\frac{6n^2-1}{12}\,w\,\delta^4 f'_0\\
w^4 f^{(4)}(a+nw)&=w\,\mu\,\delta^3 f'_0+n\,w\,\delta^4 f'_0\\
w^5 f^{(5)}(a+nw)&=w\,\delta^4 f'_0
\end{aligned}$$

Putting $n=0$ in these equations we obtain

$$\begin{aligned}
A&=wf'(a)\\
B&=\tfrac{1}{2}\,w^2 f''(a)=\tfrac{1}{2}\,w\,\mu\,\delta f'_0-\tfrac{1}{12}\,w\,\mu\,\delta^3 f'_0\\
C&=\tfrac{1}{6}\,w^3 f'''(a)=\tfrac{1}{6}\,w\,\delta^2 f'_0-\tfrac{1}{72}\,w\,\delta^4 f'_0\\
D&=\tfrac{1}{24}\,w^4 f^{(4)}(a)=\tfrac{1}{24}\,w\,\mu\,\delta^3 f'_0\\
E&=\tfrac{1}{120}\,w^5 f^{(5)}(a)=\tfrac{1}{120}\,w\,\delta^4 f'_0.
\end{aligned}$$

A third method of determining the antifunction is to determine n_1 as in the first method, and then, using shorter intervals, to retabulate the function for values of n in the neighbourhood of n_1. The second approximation is now obtained from this new table in the same way as n_1 was obtained from the original table. One or two repetitions of this process will give the value of n to a very high degree of accuracy.

An approximation to the value of the error committed by computing n by the first method will next be obtained.

Since

$$n = \frac{f - f_0}{\delta f_{\frac{1}{2}} + \dfrac{n-1}{2\,!}\,\delta^2 f_1 + \dots}$$

$$= \frac{f - f_0}{\delta f_{\frac{1}{2}} + \dfrac{n-1}{2\,!}\,w^2 f''\,(x + n'\,w)}$$

$$= \frac{f - f_0}{\delta f_{\frac{1}{2}}}\left\{1 + \frac{n-1}{2\,!}\,w^2\,\frac{f''(x+n'\,w)}{\delta f_{\frac{1}{2}}}\right\}^{-1}$$

the numerical value of the error committed is

$$|\,E\,| = \left|\frac{f - f_0}{\delta f_{\frac{1}{2}}}\cdot\frac{n-1}{2\,!}\,w^2\,\frac{f''\,(x+n'\,w)}{\delta f_{\frac{1}{2}}}\right|$$

$$= \left|\frac{n\,(n-1)}{2\,!}\,w^2\,\frac{f''\,(x+n'\,w)}{f_1 - f_0}\right|.$$

Example 1.—The following table gives the logarithms of the distance of Venus from the earth. To find when the logarithm had the value 9·9351799.

Date.			log.	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1914	Aug.	18	9·9617724					
				−74079				
		20	9·9543645		−1497			
				−75576		−46		
		22	9·9468069		−1543		3	
				−77119		−43		1
		24	9·9390950	(−77912)	−1586	(−41)	4	(1)
				−78705		−39		0
		26	9·9312245		−1625		4	
				−80330		−35		
		28	9·9231915		−1660			
				−81990				
		30	9·9149925					

We shall use the second method described above.

Let $f_0 = 9{\cdot}9390950.$

Then

$$A = -0{\cdot}0077912 + 0{\cdot}0000007 = -0{\cdot}0077905$$
$$B = -0{\cdot}0000793$$
$$C = -0{\cdot}0000007$$
$$D = 0$$
$$E = 0$$
$$n_1 = \frac{9{\cdot}9351799 - 9{\cdot}9390950}{-0{\cdot}0077905} = 0{\cdot}502535$$
$$r = -64{\cdot}5062$$
$$Br = 0{\cdot}0051153$$
$$Cr = 0{\cdot}0000452$$
$$Dr = 0$$
$$Er = 0$$
$$f_1 = -0{\cdot}0050637 \qquad f_1 n_1 = -0{\cdot}002545$$
$$f_2 = -0{\cdot}0000440 \qquad f_2 n_1^2 = -0{\cdot}000011$$
$$f_3 = 0 \qquad n_1 = 0{\cdot}502535$$
$$f_4 = 0 \qquad n = 0{\cdot}499979.$$

Hence the time required is

1914 Aug. (24 d. + 0·499979 × 48 h.)

or 1914 Aug. 24 d. 23 h. 59 min. 56 s.

Example 2.—To show that when $x = 1000$ and $w < 1$ the error in computing w from the formula

$$w = \frac{\log(x+w) - \log x}{\log(x+1) - \log x}$$

is less than 0·000125.

Here

$$|E| = \left| \frac{w(1-w)}{2\,!} \; \frac{f''(x+w')}{f_1 - f_0} \right| \text{ where } 0 < w' < 1$$
$$= \left| \frac{w(1-w)}{2} \; \frac{m}{(x+w')^2} \; \frac{1}{\log(x+1) - \log x} \right|$$

But by Taylor's theorem

$$\log(x+1) - \log x = \frac{m}{x+w''} \quad \text{where } 0 < w'' < 1.$$

$$\therefore \qquad |E| = \left| \frac{w(1-w)}{2} \; \frac{x+w''}{(x+w')^2} \right|$$
$$< \tfrac{1}{8} \cdot \tfrac{1}{1000}$$
$$< 0{\cdot}000125.$$

Example 3.—The following table gives the values of $J_1(x)$:—

x	$J_1(x)$
1·1	0·47090
1·2	0·49829
1·3	0·52202
1·4	0·54195
1·5	0·55794
1·6	0·56990
1·7	0·57777

Find the value of x corresponding to the value 0·55302 of $J_1(x)$.

21. Various Applications of Inverse Interpolation.

The method of inverse interpolation may also be employed to obtain the maximum and minimum values of a function which is given as a series of observations, or as a graph. In this case we should differentiate one of the formulae of interpolation, say,

$$f(a+nw)=f_0+n\,\delta f_{\frac{1}{2}}+\frac{n(n-1)}{2\,!}\,\delta^2 f_1+\ldots$$

Differentiation with respect to n gives

$$wf'(a+nw)=\delta f_{\frac{1}{2}}+\frac{2n-1}{2\,!}\,\delta^2 f_1+\ldots$$

For maxima or minima we have

$$f'(a+nw)=0,$$

so that n is given by the equation

$$0=\delta f_{\frac{1}{2}}+\frac{2n-1}{2}\,\delta^2 f_1+\frac{3n^2-6n+2}{6}\,\delta^3 f_{\frac{3}{2}}+\ldots$$

which is solved in the same way as the corresponding equation in last section.

When it is desired to obtain the value of n, for which the gradient of the function has a particular value c, say, the equation for determining n becomes

$$c=\delta f_{\frac{1}{2}}+\frac{2n-1}{2\,!}\,\delta^2 f_1+\ldots$$

The same method may be employed to obtain the solution of any equation involving one unknown, whether it be algebraic or transcendental. For we may tabulate the values of the expression occurring in the equation and obtain their successive differences. The solution of the equation is then the same as finding the values of the argument corresponding to zero values of the function.

Example 1.—Approximate to the roots of the equation

$$x^3+4x^2+3x-2=0.$$

First form the table.

x	$f(x)$	Δ	Δ^2	Δ^3
−3	−2			
		1·875		
−2·5	−0·125		−1·750	
		0·125		0·75
−2	0		−1·000	
		−0·875		0·75
−1·5	−0·875		−0·250	
		−1·125		0·75
−1	−2		0·500	
		−0·625		0·75
−0·5	−2·625		1·250	
		0·625		0·75
0	−2		2·000	
		2·625		0·75
0·5	0·625		2·750	
		5·375		0·75
1·0	6		3·500	
		8·875		0·75
1·5	14·875		4·250	
		13·125		0·75
2·0	28		5·000	
		18·125		0·75
2·5	46·125		5·750	
		23·875		
3·0	70			

The positions of the roots will be found by examining the column $f(x)$. We then see that when $f(x)=0$, the value of x lies between 0 and 0·5. It will also be seen that $f(x)$ is zero when $x=-2$, and that there is a possibility of another root in the interval $-2{\cdot}5 < x < -1{\cdot}5$. We shall consider the various approximations to the root for which $0 < x < 0{\cdot}5$.

Using the formulae of the first method of § 20 we have

$$n_1 = \frac{f-f_0}{\delta f_{\frac{1}{2}}}$$

$$= \frac{0-(-2)}{2{\cdot}625}$$

$$= 0{\cdot}7619$$

$$n_2 = \frac{f-f_0}{\delta f_{\frac{1}{2}} + \frac{1}{2}(n_1-1)\,\delta^2 f_1}$$

$$= \frac{2}{2{\cdot}625 - 0{\cdot}119 \times 2{\cdot}75}$$

$$= 0{\cdot}8704$$

$$n_3 = \frac{f - f_0}{\delta f_{\frac{1}{2}} + \frac{1}{2}(n_2 - 1)\,\delta^2 f_1 + \frac{1}{6}(n_2 - 1)(n_2 - 2)\,\delta^3 f_{\frac{3}{2}}}$$

$$= \frac{2}{2{\cdot}625 - 0{\cdot}0648 \times 2{\cdot}75 + 0{\cdot}0244 \times 0{\cdot}75}$$

$$= 0{\cdot}8113$$

$$n_4 = \frac{f - f_0}{\delta f_{\frac{1}{2}} + \frac{1}{2}(n_3 - 1)\,\delta^2 f_1 + \frac{1}{6}(n_3 - 1)(n_3 - 2)\,\delta^3 f_{\frac{3}{2}}}$$

$$= \frac{2}{2{\cdot}625 - 0{\cdot}0944 \times 2{\cdot}75 + 0{\cdot}0374 \times 0{\cdot}75}$$

$$= 0{\cdot}8356.$$

The first four approximations to the value of n are therefore

$$n_1 = 0{\cdot}7619$$
$$n_2 = 0{\cdot}8704$$
$$n_3 = 0{\cdot}8113$$
$$n_4 = 0{\cdot}8356$$

But the tabular interval is one-half. Hence the first four approximations to this root are

$$x_1 = 0{\cdot}3810$$
$$x_2 = 0{\cdot}4352$$
$$x_3 = 0{\cdot}4057$$
$$x_4 = 0{\cdot}4178$$

The correct value of the root is $0{\cdot}4142$. Subtracting each of the values obtained from $0{\cdot}4142$ we get

$$0{\cdot}4142 - x_1 = +0{\cdot}0332$$
$$0{\cdot}4142 - x_2 = -0{\cdot}0210$$
$$0{\cdot}4142 - x_3 = +0{\cdot}0085$$
$$0{\cdot}4142 - x_4 = -0{\cdot}0036$$

These differences show that each step gives a much closer approximation to the root, and that by continuing the process we can get a result which shall be correct to any given number of decimal places.

Examples of the second method, in which only first differences are taken into account, will be found in any text-book which discusses the solution of numerical equations.

Example 2.—Approximate to the real roots of the equations

(i) $e^{-x} - x^3 = 0$

(ii) $e^x + x^2 - 4 = 0$

(iii) $0{\cdot}5\, x^{1{\cdot}5} - 12 \log_{10} x + 2 \sin 2x = 0{\cdot}921.$

MISCELLANEOUS EXAMPLES.

1. If $y = x \tan^{-1} x$, calculate a table of first differences near $x = 0{\cdot}43$ for increments $0{\cdot}0002$ in x.

2. The following table gives the moon's right ascension

DATE.			R. A.		
1914, March 5 d.	0 h.	5 h.	3 m.	7·26 s.
	10 h.		26 m.	58·96 s.
	20 h.		51 m.	14·83 s.
6 d.	6 h.	6 h.	15 m.	48·96 s.
	16 h.		40 m.	34·76 s.
7 d.	2 h.	7 h.	5 m.	25·36 s.

Complete the table for every hour from March 5 d. 20 h. to March 6 d. 6 h.

3. Prove that in a table of logarithmic tangents to base 10, the difference for one minute in the neighbourhood of 60° is 0·00029.

4. Determine the values of (1) $\log \tan 1° \, 29' \, 33''$
(2) $\log \sin 1° \, 15' \, 12''$

5. Show that $\log_{10} 1031 = 3{\cdot}013\,258\,665\,283\,516\,546\,909\,664$.

6. Show that the number whose logarithm to 20 places of decimals is given by $0{\cdot}004\,892\,890\,303\,239\,039\,78$ is $1{\cdot}01133$.

7. From the following table determine the value of x for which $f(x)$ has the value 0·22389.

x	$f(x)$
1·5	0·51183
1·7	0·39798
1·9	0·28182
2·1	0·16661
2·3	0·05554

8. Approximate to the real roots of the equations

(i) $x e^x + 2x - 5 = 0$.

(ii) $e^x - e^{-x} + 0{\cdot}4x - 10 = 0$

(iii) $2x^{3{\cdot}1} - 3x - 16 = 0$

(iv) $2{\cdot}42 x^3 - 3{\cdot}15 \log_e x - 20{\cdot}5 = 0$.

9. From the series

$$\operatorname{ber} x = 1 - \frac{x^4}{2^2 . 4^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \frac{x^{12}}{2^2 . 4^2 . 6^2 . 8^2 . 10^2 . 12^2} + \cdots$$

compute a table of ber x from $x = 0{\cdot}0$ to $x = 5{\cdot}0$ at intervals of 0·5, correctly to 6 places of decimals.

Hence, by subtabulation, compute a table of ber x from $x = 2{\cdot}0$ to $x = 3{\cdot}0$ at intervals of 0·1.

10. The confluent hypergeometric function $W_{k,m}(z)$ is computed for large values of z from its asymptotic expansion

$$W_{k,m}(z) = e^{-\frac{1}{2}z} z^k \left\{ 1 + \sum_{n=1}^{\infty} \frac{\{m^2 - (k-\frac{1}{2})^2\}\{m^2 - (k-\frac{3}{2})^2\} \ldots \{m^2 - (k-n+\frac{1}{2})^2\}}{n!\, z^n} \right\}$$

while for small values of z it is computed from the formula

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)} M_{k,-m}(z)$$

where

$$M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} \left\{ 1 + \frac{\frac{1}{2}+m-k}{1!\,(2m+1)} z + \frac{(\frac{1}{2}+m-k)(\frac{3}{2}+m-k)}{2!\,(2m+1)(2m+2)} z^2 + \ldots \right\}.$$

Compute tables and draw graphs of $W_{k,m}(z)$ for positive values of z in the cases

$$k = -3{\cdot}1, \quad m = +2{\cdot}2$$
$$k = -0{\cdot}1, \quad m = +0{\cdot}2$$
$$k = +2{\cdot}2, \quad m = +0{\cdot}2.$$

CHAPTER IV

NUMERICAL INTEGRATION

22. Introduction.

The method of evaluating the definite integral of a function from a series of numerical values of the function is called *Mechanical Quadrature* or *Numerical Integration.*

If a function is given by its graph, the simplest way of determining an area bounded by the curve, two given ordinates and a given abscissa, is by means of a planimeter. This method would naturally be used in the case of the indicator diagrams of steam, gas, or oil engines, or the cards from certain hydraulic motors, or the stress-strain diagrams drawn by various types of testing machines. The accuracy of the result obtained would in such cases be as good as is required, but it is to be remembered that graphical methods are not susceptible of great refinement. If, therefore, considerable accuracy is required, or indeed in general whenever the function is specified by a table of numerical values, the method of numerical integration is preferable to the use of the planimeter. The method of numerical integration must also be resorted to when a function is specified by its analytical expression, but cannot be integrated in terms of known functions by the methods of the Integral Calculus; *e.g.*, the elliptic integrals belong to the class of functions which cannot be evaluated by the elementary methods of the integral calculus.

The formulae necessary for numerical integration are derived from the formulae already established for interpolation. As in the case of interpolation, the order of differences which must be taken into account, will depend entirely on the rapidity with which the differences decrease as the order increases. It need hardly be said that unless the convergence of the series can be ensured the process is of no avail. Consider, for example, the

determination of the area of a semicircle, the ordinates being perpendicular to the diameter. No matter how many ordinates are taken (*i.e.* no matter the order of differences employed), the approximation to the area by means of many of the formulae to be hereafter established will not be a good one. But if we evaluate the area included between an arc of a circle, less than a semicircle, a diameter, and the two perpendiculars from the extremities of the arc on this diameter, we can, from the same formulae, obtain an approximation to the true value with any degree of accuracy we please. The reason for the failure in the first instance is that the differential coefficients of the ordinates at the limits of integration are infinite, and as a result the series is not convergent. For this same reason we must assume that the function represented by the observations, and its differential coefficients up to the order of differences employed, are continuous; a condition which is not necessarily fulfilled in natural problems, as may be seen by an examination of an ordinary barometric curve.

23. Evaluation of Integrals by the Integration of Infinite Series Term by Term.

Before applying the methods of finite differences we shall give an example showing how in many cases the value of the integral of a function which is analytically known may be obtained by expanding the function in a uniformly convergent infinite series, and then integrating it term by term.

Example 1.—Consider the complete elliptic integral

$$F(k, \tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}}.$$

$$= \int_0^{\frac{\pi}{2}} dx\,(1 - k^2 \sin^2 x)^{-\frac{1}{2}}$$

Assuming that $|\,k\,| < 1$ and expanding the term $(1 - k^2 \sin^2 x)^{-\frac{1}{2}}$ by the binomial theorem, we have

$$F(k, \tfrac{\pi}{2}) = \int_0^{\frac{\pi}{2}} dx \left\{ 1 + \tfrac{1}{2} k^2 \sin^2 x + \frac{1\,.\,3}{2^2\,.\,2\,!} k^4 \sin^4 x \right.$$
$$\left. + \frac{1\,.\,3\,.\,5}{2^3\,.\,3\,!} k^6 \sin^6 x + \ldots \right\}.$$

But
$$\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, d\theta = \frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,.\,6\,\ldots\,2n}\,\frac{\pi}{2}.$$

Hence

$$F\left(k, \tfrac{\pi}{2}\right) = \frac{\pi}{2}\left\{1 + \left(\tfrac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2 k^6 + \ldots\right\}.$$

If k is small this series is very suitable for obtaining the value of F; but it is evident that if k approaches unity a large number of terms must be included to get a fair approximation to the value of F.

Let $k = 0{\cdot}1$. Then to obtain F correct to 4 decimal places it is only necessary to include two terms of the series. For

$$\begin{aligned} F\left(0{\cdot}1, \tfrac{\pi}{2}\right) &= \tfrac{\pi}{2}\left\{1 + \frac{0{\cdot}25}{100} + \frac{0{\cdot}140625}{10000} + \ldots\right\} \\ &= \tfrac{\pi}{2}\{1 + 0{\cdot}0025 + 0{\cdot}0000140625 + \ldots\} \\ &= \tfrac{\pi}{2}\{1{\cdot}0025\} \\ &= 1{\cdot}5747. \end{aligned}$$

Example 2.—Find the value of $F\left(k, \frac{\pi}{2}\right)$ when $k = \sin 10°$

Example 3.—Show that $\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 x}$ is equal to $1{\cdot}54415$ when k has the value $\sin 15°$.

Example 4.—Show that the value of the integral $\frac{2}{\sqrt{\pi}}\int_0^{0{\cdot}6} e^{-x^2}\,dx$ is $0{\cdot}60386$.

24. A Formula of Integration Based on Newton's Formula.

We shall now show how integration may be performed by the aid of interpolation-formulae. Integrating Newton's interpolation formula

$$f(a + nw) = f_0 + n\,\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2\,!}\,\delta^2 f_1 + \ldots$$

with respect to n, we have

$$\int f(a+nw)dn = f_0\int dn + \delta f_{\frac{1}{2}}\int n\,dn + \frac{1}{2\,!}\delta^2 f_1\int n(n-1)\,dn + \frac{1}{3\,!}\delta^3 f_{\frac{3}{2}}\int n(n-1)(n-2)\,dn + \dots$$

$$= f_0 \,.\, n + \delta f_{\frac{1}{2}} \,.\, \frac{n^2}{2} + \frac{1}{2\,!}\delta^2 f_1\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \frac{1}{3\,!}\delta^3 f_{\frac{3}{2}}\left(\frac{n^4}{4} - n^3 + n^2\right) + \dots$$

Hence

$$\int_0^1 f(a+nw)\,dn = f_0 + \tfrac{1}{2}\,\delta f_{\frac{1}{2}} - \tfrac{1}{12}\,\delta^2 f_1 + \tfrac{1}{24}\,\delta^3 f_{\frac{3}{2}} - \tfrac{19}{720}\,\delta^4 f_2 + \dots$$

$$\int_1^2 f(a+nw)\,dn = f_1 + \tfrac{1}{2}\,\delta f_{\frac{3}{2}} - \tfrac{1}{12}\,\delta^2 f_2 + \tfrac{1}{24}\,\delta^3 f_{\frac{5}{2}} - \tfrac{19}{720}\,\delta^4 f_3 + \dots$$

$$\int_2^3 f(a+nw)\,dn = f_2 + \tfrac{1}{2}\,\delta f_{\frac{5}{2}} - \tfrac{1}{12}\,\delta^2 f_3 + \tfrac{1}{24}\,\delta^3 f_{\frac{7}{2}} - \tfrac{19}{720}\,\delta^4 f_4 + \dots$$

....

$$\int_{r-1}^r f(a+nw)\,dn = f_{r-1} + \tfrac{1}{2}\,\delta f_{r-\frac{1}{2}} - \tfrac{1}{12}\,\delta^2 f_r + \tfrac{1}{24}\,\delta^3 f_{r+\frac{1}{2}} - \tfrac{19}{720}\,\delta^4 f_{r+1} + \dots$$

Adding these equations we have

$$\begin{aligned}\int_0^r f(a+nw)\,dn = {} & f_0 + f_1 + \dots + f_{r-1} \\ & + \tfrac{1}{2}\left(\delta f_{\frac{1}{2}} + \delta f_{\frac{3}{2}} + \dots + \delta f_{r-\frac{1}{2}}\right) \\ & - \tfrac{1}{12}\left(\delta^2 f_1 + \delta^2 f_2 + \dots + \delta^2 f_r\right) \\ & + \tfrac{1}{24}\left(\delta^3 f_{\frac{3}{2}} + \delta^3 f_{\frac{5}{2}} + \dots + \delta^3 f_{r+\frac{1}{2}}\right) \\ & - \tfrac{19}{720}\left(\delta^4 f_2 + \delta^4 f_3 + \dots + \delta^4 f_{r+1}\right) \\ & + \dots\dots\dots\dots\end{aligned}$$

But

$$\begin{aligned}\delta f_{\frac{1}{2}} + \delta f_{\frac{3}{2}} + \dots + \delta f_{r-\frac{1}{2}} &= f_1 - f_0 + f_2 - f_1 + \dots + f_r - f_{r-1} \\ &= f_r - f_0\end{aligned}$$

$$\begin{aligned}\delta^2 f_1 + \delta^2 f_2 + \dots\ \delta^2 f_r &= \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}} + \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}} + \dots + \delta f_{r+\frac{1}{2}} - \delta f_{r-\frac{1}{2}} \\ &= \delta f_{r+\frac{1}{2}} - \delta f_{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\delta^3 f_{\frac{3}{2}} + \delta^3 f_{\frac{5}{2}} + \dots + \delta^3 f_{r+\frac{1}{2}} &= \delta^2 f_2 - \delta^2 f_1 + \delta^2 f_3 - \delta^2 f_2 + \dots + \delta^2 f_{r+1} - \delta^2 f_r \\ &= \delta^2 f_{r+1} - \delta^2 f_1\end{aligned}$$

$$\begin{aligned}\delta^4 f_2 + \delta^4 f_3 + \dots + \delta^4 f_{r+1} &= \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}} + \delta^3 f_{\frac{7}{2}} - \delta^3 f_{\frac{5}{2}} + \dots + \delta^3 f_{r+\frac{3}{2}} - \delta^3 f_{r+\frac{1}{2}} \\ &= \delta^3 f_{r+\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}\end{aligned}$$

Hence

$$\int_0^r f(a+nw)\,dn = \tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{r-1} + \tfrac{1}{2}f_r$$
$$- \tfrac{1}{12}(\delta f_{r+\frac{1}{2}} - \delta f_{\frac{1}{2}}) + \tfrac{1}{24}(\delta^2 f_{r+1} - \delta^2 f_1)$$
$$- \tfrac{19}{720}(\delta^3 f_{r+\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}) + \dots\dots\dots\dots\dots$$

Putting $x = a + nw$ this gives

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = \tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{r-1} + \tfrac{1}{2}f_r$$
$$- \tfrac{1}{12}(\delta f_{r+\frac{1}{2}} - \delta f_{\frac{1}{2}}) + \tfrac{1}{24}(\delta^2 f_{r+1} - \delta^2 f_1)$$
$$- \tfrac{19}{720}(\delta^3 f_{r+\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}) + \dots\dots\dots\dots\dots$$

This expansion will give the value of the definite integral, provided it is possible to obtain values of $f(x)$ outside the limits of integration in order to obtain the differences.

When this result is written in the form

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = (f_1 + f_2 + \dots + f_{r-1}) + \tfrac{1}{2}(f_r + f_0) - \tfrac{1}{12}(\delta f_{r+\frac{1}{2}} - \delta f_{\frac{1}{2}})$$
$$+ \tfrac{1}{24}(\delta^2 f_{r+1} - \delta^2 f_1) - \tfrac{19}{720}(\delta^3 f_{r+\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}) + \dots$$

it is easily seen that the coefficients are those of x in the expansion

$$\frac{x}{\log(1+x)} = 1 + A_0 x + A_1 x^2 + A_2 x^3 + \dots$$

Clausen * has computed the first thirteen of these, viz. :—

$$A_0 = +\frac{1}{2} \qquad A_7 = -\frac{33953}{3628800}$$
$$A_1 = -\frac{1}{12} \qquad A_8 = +\frac{8183}{1036800}$$
$$A_2 = +\frac{1}{24} \qquad A_9 = -\frac{3250433}{479001600}$$
$$A_3 = -\frac{19}{720} \qquad A_{10} = +\frac{4671}{788480}$$
$$A_4 = +\frac{3}{160} \qquad A_{11} = -\frac{13695779093}{2615348736000}$$
$$A_5 = -\frac{863}{60480} \qquad A_{12} = +\frac{2224234463}{475517952000}$$
$$A_6 = +\frac{275}{24192}$$

* *Jour. reine angew. Math.*, 6 (1830), p. 287–9.

In the same memoir it is shown that when the limits of integration are $a-\frac{1}{2}w$ and $a+(r+\frac{1}{2})w$ the coefficients are those in the expansion

$$\frac{x}{\sqrt{(1+x)}\log(1+x)} = 1 + B_1x^2 + B_2x^3 + B_3x^4 + \dots$$

These are also tabulated.

The formula obtained above must be subjected to another transformation when values outside the limits of integration cannot be obtained.

Now

(i) $\quad \delta f_{r+\frac{1}{2}} = \delta^2 f_r + \delta f_{r-\frac{1}{2}}$

(ii) $\quad \delta^2 f_{r+1} - 2\,\delta^2 f_r + \delta^2 f_{r-1} = \delta^4 f_r$

i.e. $\quad \delta^2 f_{r+1} - 2\,\delta^2 f_r = -\,\delta^2 f_{r-1} + \delta^4 f_r$

(iii) $\quad \delta^6 f_r = \delta^3 f_{r+\frac{3}{2}} - 3\,\delta^3 f_{r+\frac{1}{2}} + 3\,\delta^3 f_{r-\frac{1}{2}} - \delta^3 f_{r-\frac{3}{2}}$

i.e. $\quad \delta^3 f_{r+\frac{3}{2}} = \delta^6 f_r + 3\,\delta^3 f_{r+\frac{1}{2}} - 3\,\delta^3 f_{r-\frac{1}{2}} + \delta^3 f_{r-\frac{3}{2}}$

Substituting these values in the formula we have

$$\begin{aligned}\frac{1}{w}\int_a^{a+rw} f(x)\,dx &= \tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{r-1} + \tfrac{1}{2}f_r \\ &\quad - \tfrac{1}{12}(\delta f_{r-\frac{1}{2}} - \delta f_{\frac{1}{2}}) - \tfrac{1}{24}(\delta^2 f_{r-1} + \delta^2 f_1) \\ &\quad - \tfrac{19}{720}(\delta^3 f_{r-\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}) - \tfrac{3}{160}(\delta^4 f_{r-2} + \delta^4 f_2) \\ &\quad \dots\dots\dots\dots\end{aligned}$$

This form involves only differences which can be calculated from values of the function which are within the limits of integration, and is the form to be used when no values outside those limits can be obtained.

Example 1.—To determine π from the equation

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

using differences of $\frac{1}{10}$.

Here $\quad f(x) = \dfrac{1}{1+x^2}\,; \quad f_0 = f(0) = 1\,; \quad f_r = f(1) = 0{\cdot}5\,;$

$$w = 0{\cdot}1\,; \quad r = 10.$$

Inspection of the formula will show that it is not necessary to form a complete table of differences. The highest order of differences retained will be the 4th, and as $\delta^4 f_2$ and $\delta^4 f_8$ are the only two of this order required, the central portion of the table may be omitted as in the following scheme :—

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0·0	1·00000				
		− 990			
0·1	0·99010		− 1866		
		− 2856		311	
0·2	0·96154		− 1555		119
		− 4411		430	
0·3	0·91743		− 1125		
		− 5536			
0·4	0·86207				
0·5	0·80000				
0·6	0·73529				
		− 6415			
0·7	0·67114		277		
		− 6138		134	
0·8	0·60976		411		− 67
		− 5727		67	
0·9	0·55249		478		
		− 5249			
1·0	0·50000				

From this table we have

$$\begin{aligned}
\tfrac{1}{2}f_0 &= 0{\cdot}50000 \\
f_1 &= 0{\cdot}99010 \\
f_2 &= 0{\cdot}96154 \\
f_3 &= 0{\cdot}91743 \\
f_4 &= 0{\cdot}86207 \\
f_5 &= 0{\cdot}80000 \\
f_6 &= 0{\cdot}73529 \\
f_7 &= 0{\cdot}67114 \\
f_8 &= 0{\cdot}60976 \\
f_9 &= 0{\cdot}55249 \\
\tfrac{1}{2}f_{10} &= 0{\cdot}25000 \\
\hline
\text{Sum} &= 7{\cdot}84982
\end{aligned}$$

$$\begin{aligned}
\delta f_{9\frac{1}{2}} &= -0{\cdot}05249 \\
\delta f_{\frac{1}{2}} &= -0{\cdot}00990 \\
\hline
\text{Diff.} &= -0{\cdot}04259 \\
\hline
\delta^2 f_9 &= 0{\cdot}00478 \\
\delta^2 f_1 &= -0{\cdot}01866 \\
\hline
\text{Sum} &= -0{\cdot}01388 \\
\hline
\delta^3 f_{8\frac{1}{2}} &= 0{\cdot}00067 \\
\delta^3 f_{1\frac{1}{2}} &= 0{\cdot}00311 \\
\hline
\text{Diff.} &= -0{\cdot}00244 \\
\hline
\delta^4 f_8 &= -0{\cdot}00067 \\
\delta^4 f_2 &= 0{\cdot}00119 \\
\hline
\text{Sum} &= 0{\cdot}00052 \\
\hline
\end{aligned}$$

Hence

$$10\int_0^1 \frac{dx}{1+x^2} = 7{\cdot}84982 - \frac{1}{12}(-0{\cdot}04259) - \frac{1}{24}(-0{\cdot}01388)$$
$$- \frac{19}{720}(-0{\cdot}00244) \quad \frac{3}{160}\,(0{\cdot}00052)$$

$$
\begin{aligned}
&= 7{\cdot}84982 \\
&\qquad\quad 355 \\
&\qquad\quad\; 58 \\
&\qquad\quad 6 - 1 \\
&= 7{\cdot}85400
\end{aligned}
$$

$$\therefore \quad \frac{10\,\pi}{4} = 7{\cdot}85400$$

$$\pi = 3{\cdot}1416\dot{0}.$$

Example 2.—Determine $\log_e 2$ from the equation $\frac{1}{2}\log_e 2 = \int_2^3 \frac{x\,dx}{1+x^2}$ using differences of $\frac{1}{10}$.

25. Case when the Upper Limit is not a Tabulated Value.

In the previous section it has been assumed that the quantity r in the upper limit was an integer, so that the tabular value of $f(x)$ was known when x had the value $a+r\,w$. When the upper limit is $a+r'\,w$, where $r<r'<r+1$, and the lower limit coincides with one of the values of the argument, so that $f(x)$ is known when $x=a,\ a+w,\ \ldots\ a+r\,w,\ a+\overline{r+1}\,w,\ \ldots$ the value of the integral may be obtained as follows :—

First determine, as in last section, the value of

$$\int_a^{a+r\,w} f(x)\,d\,x.$$

Then, dividing the range $(a+r\,w,\ a+r'\,w)$ into a convenient number of equal intervals, interpolate the values of $f(x)$ for each of these sub-intervals. The value of

$$\int_{a+r\,w}^{a+r'\,w} f(x)\,dx$$

may now be obtained by making use of these new values of $f(x)$ and their differences.

When the lower limit does not coincide with a tabular value of the argument, a similar process will enable the value of the integral between the given limit and the nearest tabular value of the argument to be obtained.

Another method which might be employed is to obtain the value of $\int f(x)\,dx$ for different ranges of integration, and then determine the value for the given range by interpolation.

Example 1.—Evaluate $\int_{20^\circ}^{31^\circ 20'} \cos x\,dx$, given the values of $\cos x$ when $x = 20^\circ, 22^\circ, 24^\circ, 26^\circ, 28^\circ, 30^\circ, 32^\circ$.

Forming the table of differences we have

x	$\cos x$	Δ	Δ^2	Δ^3	Δ^4
20°	0·9396926				
		−125087			
22°	0·9271839		−11297		
		−136384		166	
24°	0·9135455		−11131		16
		−147515		182	
26°	0·8987940		−10949		9
		−158464		191	
28°	0·8829476		−10758		16
		−169222		207	
30°	0·8660254		−10551		
		−179773			
32	0·8480481				

First Method.—We shall first of all determine $\int_{20^\circ}^{30^\circ} \cos x\,dx$. Substituting in the formula

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = \tfrac{1}{2}f_0 + f_1 + \dots + f_{r-1} + \tfrac{1}{2}f_r - \tfrac{1}{12}(\delta f_{r-\frac{1}{2}} - \delta f_{\frac{1}{2}}) - \tfrac{1}{24}(\delta^2 f_{r-1} + \delta^2 f_1)$$
$$- \tfrac{19}{720}(\delta^3 f_{r-\frac{3}{2}} - \delta^3 f_{\frac{3}{2}}) - \tfrac{3}{160}(\delta^4 f_{r-2} + \delta^4 f_2)$$

we have

$$\frac{1}{0{\cdot}0349066}\int_{20^\circ}^{30^\circ} \cos x\,dx = \begin{array}{r} 0{\cdot}4698463 \\ 0{\cdot}9271839 \\ 0{\cdot}9135455 \\ 0{\cdot}8987940 \\ 0{\cdot}8829476 \\ 0{\cdot}4330127 \end{array} - \tfrac{1}{12}\begin{bmatrix} -169222 \\ +125087 \end{bmatrix} - \tfrac{1}{24}\begin{bmatrix} -10758 \\ -11297 \end{bmatrix} - \tfrac{19}{720}\begin{bmatrix} 191 \\ -166 \end{bmatrix}$$

$$= 4{\cdot}5257896$$

$$\int_{20^\circ}^{30^\circ} \cos x\,dx = 0{\cdot}1579798.$$

Now by using Newton's formula for backward interpolation

$$f(a - nw) = f_0 - n\,\delta f_{-\frac{1}{2}} + \frac{n(n-1)}{2\,!}\,\delta^2 f_{-1} - \frac{n(n-1)(n-2)}{3\,!}\,\delta^3 f_{-\frac{3}{2}} + \dots$$

determine the values of $f(x)$, *i.e.* $\cos x$, when x has the values 30° 20′, 30° 40′, 31°, 31° 20′, 31° 40′. We thus obtain the following table :—

x	$\cos x$	Δ	Δ^2	Δ^3
30° 0′	0·8660254			
		−29235		
30° 20′	0·8631019		−293	
		−29528		3
30° 40′	0·8601491		−290	
		−29818		1
31° 0′	0·8571673		−291	
		−30109		3
31° 20′	0·8541564		−288	
		−30397		1
31° 40′	0·8511167		−289	
		−30686		
32° 0′	0·8480481			

From this table we obtain

$$\begin{aligned}\frac{1}{0\cdot0058178}\int_{30}^{31^\circ 20'}\cos x\,dx = {} & 0\cdot4330127 - \tfrac{1}{12}\begin{bmatrix}-30109\\+29235\end{bmatrix} - \tfrac{1}{24}\begin{bmatrix}-291\\-293\end{bmatrix}\\ & 0\cdot8631019\\ & 0\cdot8601491\\ & 0\cdot8571673\\ & 0\cdot4270782\\ = {} & 3\cdot4405189\end{aligned}$$

$$\therefore \int_{30}^{31^\circ 20'}\cos x\,dx = 0\cdot0200161$$

Adding these results together we get

$$\int_{20^\circ}^{31^\circ 20'}\cos x\,dx = 0\cdot1779959$$

Second Method.—Obtain by the general method the values of $\int^x \cos x\,dx$ when the lower limit is 20° and the upper limits are 22°, 24°, 26°, 28°, 30°, 32° respectively. We thus obtain the following table:—

x	$\int^x \cos x\,dx$	Δ	Δ^2	Δ^3
20°	0·0000000			
		325865		
22°	0·0325865		−4565	
		321300		−390
24°	0·0647165		−4955	
		316345		−385
26°	0·0963510		−5340	
		311005		−381
28°	0·1274515		−5721	
		305284		−370
30°	0·1579799		−6091	
		299193		
32°	0·1878992			

Using Newton's formula of backward interpolation we obtain the value of the required integral, viz.

$$\int_{20^\circ}^{31^\circ 20'} \cos x\, dx = 0{\cdot}1878992 - \tfrac{1}{3}(299193) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2\,!}(-6091) - \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3\,!}(-370)$$
$$= 0{\cdot}1779961.$$

The values obtained by the two methods agree to 6 places of decimals. The result correct to 8 places is 0·17799598, but as the last figure is always forced, the one method can hardly be assumed better than the other. Probably the latter method is to be preferred, as in the former the division of the range between the given upper limit and the new upper limit into suitable intervals may give rise to values of the argument consisting of many decimal figures which would increase the amount of labour involved in the interpolation.

Example 2.—Evaluate $\int_{20^\circ}^{30^\circ 20'} \sin x\, dx$, given the values of $\sin x$ when $x = 20^\circ, 22^\circ, 24^\circ, 26^\circ, 28^\circ, 30^\circ, 32^\circ$.

Example 3.—Evaluate $\int_{1\cdot5}^{4\cdot5} \frac{dx}{x+10}$ if the values of $\frac{1}{x+10}$ are given for integral values of x.

26. A Formula Based on Bessel's Expansion.

Another series for the definite integral is obtained by integrating Bessel's formula of interpolation,

$$f(a+nw) = \mu f'_{\frac{1}{2}} + (n-\tfrac{1}{2})\,\delta f'_{\frac{1}{2}} + \frac{n(n-1)}{2\,!}\,\mu\,\delta^2 f'_{\frac{1}{2}} + \frac{n(n-1)(n-\frac{1}{2})}{3\,!}\,\delta^3 f'_{\frac{1}{2}}$$
$$+ \frac{n(n^2-1)(n-2)}{4\,!}\,\mu\,\delta^4 f'_{\frac{1}{2}} + \dots$$

Putting $x = a + nw$ we have by integration

$$\frac{1}{w}\int_a^{a+w} f(x)\,dx = \int_0^1 f(a+nw)\,dn$$
$$= \mu f'_{\frac{1}{2}}\int_0^1 dn + \delta f'_{\frac{1}{2}}\int_0^1 (n-\tfrac{1}{2})\,dn + \mu\,\delta^2 f'_{\frac{1}{2}}\int_0^1 \frac{n(n-1)}{2\,!}\,dn$$
$$+ \delta^3 f'_{\frac{1}{2}}\int_0^1 \frac{n(n-1)(n-\frac{1}{2})}{3\,!}\,dn$$
$$+ \mu\,\delta^4 f'_{\frac{1}{2}}\int_0^1 \frac{n(n^2-1)(n-2)}{4\,!}\,dn + \dots$$
$$= \mu f'_{\frac{1}{2}} - \tfrac{1}{12}\,\mu\,\delta^2 f'_{\frac{1}{2}} + \tfrac{11}{720}\,\mu\,\delta^4 f'_{\frac{1}{2}} + \dots$$

Similarly,

$$\frac{1}{w}\int_{a+w}^{a+2w} f(x)\,dx = \mu f_{\frac{3}{2}} - \tfrac{1}{12}\mu\,\delta^2 f_{\frac{3}{2}} + \tfrac{11}{720}\mu\,\delta^4 f_{\frac{3}{2}} + \dots$$

....

$$\frac{1}{w}\int_{a+(r-1)w}^{a+rw} f(x)\,dx = \mu f_{r-\frac{1}{2}} - \tfrac{1}{12}\mu\,\delta^2 f_{r-\frac{1}{2}} + \tfrac{11}{720}\mu\,\delta^4 f_{r-\frac{1}{2}} + \dots$$

Hence

$$\begin{aligned}\frac{1}{w}\int_{a}^{a+rw} f(x)\,dx &= \mu f_{\frac{1}{2}} + \mu f_{\frac{3}{2}} + \dots + \mu f_{r-\frac{1}{2}}\\ &\quad - \tfrac{1}{12}\{\mu\,\delta^2 f_{\frac{1}{2}} + \mu\,\delta^2 f_{\frac{3}{2}} + \dots + \mu\,\delta^2 f_{r-\frac{1}{2}}\}\\ &\quad + \tfrac{11}{720}\{\mu\,\delta^4 f_{\frac{1}{2}} + \mu\,\delta^4 f_{\frac{3}{2}} + \dots + \mu\,\delta^4 f_{r-\frac{1}{2}}\}\\ &\quad + \dots\dots\dots\dots\\ &= \tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{r-1} + \tfrac{1}{2}f_r\\ &\quad - \tfrac{1}{12}\{\tfrac{1}{2}\delta^2 f_0 + \delta^2 f_1 + \delta^2 f_2 + \dots + \delta^2 f_{r-1} + \tfrac{1}{2}\delta^2 f_r\}\\ &\quad + \tfrac{11}{720}\{\tfrac{1}{2}\delta^4 f_0 + \delta^4 f_1 + \delta^4 f_2 + \dots + \delta^4 f_{r-1} + \tfrac{1}{2}\delta^4 f_r\}\\ &\quad + \dots\dots\dots\dots\end{aligned}$$

or, since

$$\begin{aligned}\delta^2 f_1 + \delta^2 f_2 + \dots + \delta^2 f_{r-1} &= \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}} + \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}} + \dots + \delta f_{r-\frac{1}{2}} - \delta f_{r-\frac{3}{2}}\\ &= \delta f_{r-\frac{1}{2}} - \delta f_{\frac{1}{2}}\end{aligned}$$

and therefore

$$\delta^4 f_1 + \delta^4 f_2 + \dots + \delta^4 f_{r-1} = \delta^3 f_{r-\frac{1}{2}} - \delta^3 f_{\frac{1}{2}}$$

we have

$$\begin{aligned}\frac{1}{w}\int_{a}^{a+rw} f(x)\,dx &= \tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{r-1} + \tfrac{1}{2}f_r\\ &\quad - \tfrac{1}{12}\{\tfrac{1}{2}\delta^2 f_0 - \delta f_{\frac{1}{2}} + \delta f_{r-\frac{1}{2}} + \tfrac{1}{2}\delta^2 f_r\}\\ &\quad + \tfrac{11}{720}\{\tfrac{1}{2}\delta^4 f_0 - \delta^3 f_{\frac{1}{2}} + \delta^3 f_{r-\frac{1}{2}} + \tfrac{1}{2}\delta^4 f_r\}\\ &\quad + \dots\dots\dots\dots\end{aligned}$$

But

$$\begin{aligned}\tfrac{1}{2}\delta^2 f_0 - \delta f_{\frac{1}{2}} &= \tfrac{1}{2}\delta f_{\frac{1}{2}} - \tfrac{1}{2}\delta f_{-\frac{1}{2}} - \delta f_{\frac{1}{2}}\\ &= -\tfrac{1}{2}(\delta f_{-\frac{1}{2}} + \delta f_{\frac{1}{2}})\\ &= -\mu\,\delta f_0\\ \tfrac{1}{2}\delta^4 f_0 - \delta^3 f_{\frac{1}{2}} &= -\mu\,\delta^3 f_0\end{aligned}$$

and, similarly,

$$\tfrac{1}{2}\delta^2 f_r + \delta f_{r-\frac{1}{2}} = \mu\,\delta f_r$$

$$\tfrac{1}{2}\delta^4 f_r + \delta^3 f_{r-\frac{1}{2}} = \mu\,\delta^3 f_r.$$

Hence, finally, we obtain the formula

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = \tfrac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{r-1} + \tfrac{1}{2}f_r$$
$$- \tfrac{1}{12}(\mu\,\delta f_r - \mu\,\delta f_0) + \tfrac{11}{720}(\mu\,\delta^3 f_r - \mu\,\delta^3 f_0)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Example 1.—Calculate $\int_{100}^{105}\frac{dx}{x}$ to 8 places of decimals.

Here $f(x) = \frac{1}{x}$, $f_0 = 100$, $r = 5$, $w = 1$.

In order to obtain the differences required, it will be necessary to form $f(x)$ for all integral values of x from $x=98$ to $x=107$. These may be obtained from Barlow's Tables. We thus obtain the following :—

x	$f(x)$	Δ	Δ^2	Δ^3
98	0·010204082			
		−103072		
99	0·010101010		2062	
		−101010		−62
100	0·010000000	(−100010)	2000	(−60)
		− 99010		−58
101	0·009900990		1942	
		− 97068		−58
102	0·009803922		1884	
		− 95184		−53
103	0·009708738		1831	
		− 93353		−53
104	0·009615385		1778	
		− 91575		−51
105	0·009523810	(− 90712)	1727	(−49)
		89848		−47
106	0·009433962		1680	
		− 88168		
107	0·009345794			

Substituting in the formula

$$\int_{100}^{105}\frac{dx}{x} = \tfrac{1}{2}f_0 + f_1 + f_2 + f_3 + f_4 + \tfrac{1}{2}f_5 - \tfrac{1}{12}(\mu\,\delta f_5 - \mu\,\delta f_0) + \tfrac{11}{720}(\mu\,\delta^3 f_5 - \mu\,\delta^3 f_0)$$

we get

$$\int_{100}^{105}\frac{dx}{x} = 0{\cdot}048790940 - 0{\cdot}000000775 + 0{\cdot}000000000$$
$$= 0{\cdot}048790165.$$

The value correct to 10 places is 0·0487901642.

Note.—It is to be remarked that the expression

$$\tfrac{1}{2}f_0+f_1+f_2+f_3+f_4+\tfrac{1}{2}f_5=0{\cdot}048790940$$

gives the answer correct to six decimal places; this approximation is quite sufficient for most practical purposes.

Example 2.—Approximate to the value of the integral $\displaystyle\int_0^2 \frac{dx}{\sqrt{(x^3-2x^2+2)}}$.

27. The Euler-Maclaurin Expansion.

In the foregoing the value of the integral is expressed as a series involving differences. In many cases in which the function is known analytically, it will be found more convenient to use what is known as the *Euler-Maclaurin* expansion which employs differential coefficients instead of differences. The rapidity of convergence is of the same order as that given by Bessel's formula. It may be obtained as follows:—

Since

$$\begin{aligned}
\mu\,\delta f_0 &= \tfrac{1}{2}(f_1-f_{-1})\\
&= \tfrac{1}{2}\{f(a+w)-f(a-w)\}\\
&= \tfrac{1}{2}\{f(a)+wf'(a)+\frac{w^2}{2\,!}f''(a)+\ldots\\
&\qquad -f(a)+wf'(a)-\frac{w^2}{2\,!}f''(a)+\ldots\}\\
&= wf'(a)+\frac{w^3}{3\,!}f'''(a)+\ldots
\end{aligned}$$

and similarly

$$\mu\,\delta^3 f_0 = w^3 f'''(a)+\frac{w^5}{4}f^{(v)}(a)+\ldots$$

$$\mu\,\delta f_r = wf'(a+rw)+\frac{w^3}{3\,!}f'''(a+rw)+\ldots$$

$$\mu\,\delta^3 f_r = w^3 f'''(a+rw)+\frac{w^5}{4}f^{(v)}(a+rw)+\ldots$$

we have by substituting in the formula

$$\begin{aligned}
\frac{1}{w}\int_a^{a+rw} f(x)\,dx &= \tfrac{1}{2}f_0+f_1+f_2+\ldots+f_{r-1}+\tfrac{1}{2}f_r\\
&\quad -\tfrac{1}{12}\{\mu\,\delta f_r-\mu\,\delta f_0\}+\tfrac{11}{720}\{\mu\,\delta^3 f_r-\mu\,\delta^3 f_0\}\\
&\quad -\tfrac{191}{60480}(\mu\,\delta^5 f_r-\mu\,\delta^5 f_0)+\ldots
\end{aligned}$$

the result

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = \tfrac{1}{2}f_0 + f_1 + \dots + f_{r-1} + \tfrac{1}{2}f_r$$
$$- \tfrac{1}{12}w(f_r' - f_0') + \tfrac{1}{720}w^3(f_r''' - f_0''')$$
$$- \tfrac{1}{30240}w^5(f_r^{(v)} - f_0^{(v)}) + \dots$$

This is the *Euler-Maclaurin** expansion. The coefficients of the various items in it are proportional to the well-known Bernoullian numbers. For if † we put $f(x) = e^{x-a}$ in the formula and at the same time change the limits to $a - w$ and $a + w$ we get

$$\frac{1}{w}\int_{a-w}^{a+w} e^{x-a}\,dx = \tfrac{1}{2}f_0 + f_1 + \tfrac{1}{2}f_2 + A\,w(f_2' - f_1') + B\,w^3(f_2''' - f_0''') + \dots$$

where A, B, ... are constants.

But

$$\frac{1}{w}\int_{a-w}^{a+w} e^{x-a}\,dx = \frac{1}{w}\int_{-w}^{w} e^x\,dx$$
$$= \frac{1}{w}(e^w - e^{-w}).$$

Hence

$$\frac{1}{w}(e^w - e^{-w}) = \tfrac{1}{2}f_0 + f_1 + \tfrac{1}{2}f_2 + A\,w(f_2' - f_0') + B\,w^3(f_2''' - f_0''') + \dots$$
$$= \tfrac{1}{2}e^{-w} + 1 + \tfrac{1}{2}e^w + A\,w(e^w - e^{-w}) + B\,w^3(e^w - e^{-w}) + \dots$$

from which it is easy to deduce that

$$\frac{w}{2}\,\frac{e^{w/2} + e^{-w/2}}{e^{w/2} - e^{-w/2}} = 1 - A\,w^2 - B\,w^4 - \dots$$

or

$$\frac{w}{2i}\cot\frac{w}{2i} = 1 - A\,w^2 - B\,w^4 - \dots$$

But the Bernoullian numbers are the quantities B_1, B_2, ... in the expansion

$$\frac{z}{2}\cot\frac{z}{2} = 1 - B_1\frac{z^2}{2\,!} - B_2\frac{z^4}{4\,!} - B_3\frac{z^6}{6\,!} - \dots$$

and therefore A, B, ... are proportional to B_1, B_2, ...

---

* For a rigorous discussion, *cf.* Whittaker and Watson's *Modern Analysis*, p. 128.

† Poisson, *Mém. de l'Acad. des Sc.*, 1823.

Example 1.—Calculate $\int_{95}^{100} \frac{dx}{x}$ to 8 places of decimals, and check the result by computing it as $-\log_e(1-\frac{5}{100})$ by the logarithmic series.

Here $f_0=\frac{1}{95}$, $w=1$, $r=5$.

Then

$$\int_{95}^{100} \frac{dx}{x} = \frac{1}{2}\cdot\frac{1}{95}+\frac{1}{96}+\frac{1}{97}+\frac{1}{98}+\frac{1}{99}+\frac{1}{2}\cdot\frac{1}{100}$$

$$-\tfrac{1}{12}\left(-\frac{1}{100^2}+\frac{1}{95^2}\right)+\tfrac{1}{120}\left(-\frac{1}{100^4}+\frac{1}{95^4}\right)$$

$$\begin{aligned}
= \ & 0{\cdot}005263158 - \tfrac{1}{12}\begin{pmatrix}-0{\cdot}000100000\\ +0{\cdot}000110803\end{pmatrix}+\tfrac{1}{120}\begin{pmatrix}-0{\cdot}000000010\\ +0{\cdot}000000012\end{pmatrix}\\
& 0{\cdot}010416667\\
& 0{\cdot}010309278\\
& 0{\cdot}010204082\\
& 0{\cdot}010101010\\
& 0{\cdot}005000000\\
= \ & 0{\cdot}051294195\\
& -0{\cdot}000000900\\
= \ & 0{\cdot}051293295
\end{aligned}$$

Again

$$\begin{aligned}
-\log_e\left(1-\frac{5}{100}\right) = \ & 0{\cdot}05+\frac{(0{\cdot}05)^2}{2}+\frac{(0{\cdot}05)^3}{3}+\frac{(0{\cdot}05)^4}{4}+\frac{(0{\cdot}05)^5}{5}+\frac{(0{\cdot}05)^6}{6}\\
= \ & 0{\cdot}050000000\\
& 0{\cdot}001250000\\
& 0{\cdot}000041667\\
& 0{\cdot}000001563\\
& 0{\cdot}000000063\\
& 0{\cdot}000000003\\
= \ & 0{\cdot}051293296
\end{aligned}$$

The two results will be seen to agree to 8 places of decimals.

Example 2.—Evaluate $\int_{100}^{105} \frac{dx}{x}$ to 8 places of decimals.

Example 3.—Approximate to the value of $\int_0^{\frac{\pi}{4}} \cos^{\frac{1}{2}} x\, dx$.

28. An Exceptional Case.

Legendre* remarked that if the odd differential coefficients above a certain order take the same value at both limits, the formula fails to give an accurate value of the definite integral. For instance, if the integral under

* *Fonctions Elliptiques*, Vol. II., p. 57.

discussion were the elliptic integral of the second kind taken between limits 0 and $\pi/2$, viz.

$$\int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}\, dx,$$

all the odd differential coefficients become zero at both limits, and thus the value of the integral would appear to be given by

$$\tfrac{1}{2}f_0+f_1+f_2+\ldots+f_{r-1}+\tfrac{1}{2}f_r .$$

But the value given by this expression is not in close agreement with the true value of the integral. The reason of this is that the numerical coefficients introduced in the successive differentiations increase without limit, so that each term takes the form $\infty \times 0$, which is, of course, indeterminate.

Better methods of evaluating the three kinds of elliptic integrals will be given in another tract in the present series.

29. The Formulae of Woolhouse and Lubbock.

We shall next derive from the Euler-Maclaurin expansion a formula which is of use when it is required to evaluate the sum of a large number of terms.

If the interval between consecutive values of the argument in the Euler-Maclaurin formula is w, we have

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = f_0+f_1+f_2+\ldots+f_r-\tfrac{1}{2}(f_0+f_r)-\tfrac{1}{12}w(f_r'-f_0') + \tfrac{1}{720}w^3(f_r'''-f_0''')+\ldots$$

and if this interval is w/m the formula becomes

$$\frac{m}{w}\int_a^{a+rw} f(x)\,dx = f_0+f_{\frac{1}{m}}+f_{\frac{2}{m}}+\ldots+f_r-\tfrac{1}{2}(f_0+f_r) - \tfrac{1}{12}\frac{w}{m}(f_r'-f_0')+\tfrac{1}{720}\frac{w^3}{m^3}(f_r'''-f_0''')+\ldots$$

Subtracting the latter from m times the former we deduce

$$f_0+f_{\frac{1}{m}}+f_{\frac{2}{m}}+\ldots+f_r = m(f_0+f_1+\ldots+f_r)-\frac{m-1}{2}(f_0+f_r) - \frac{m^2-1}{12}\frac{w}{m}(f_r'-f_0') + \frac{m^4-1}{720}\frac{w^3}{m^3}(f_r'''-f_0''')+\ldots$$

This is *Woolhouse's Formula.* Obviously by using the right-hand expression instead of the left-hand one, the labour involved in determining the sum $f_0+f_{\frac{1}{m}}+f_{\frac{2}{m}}+\dots+f_r$ is enormously reduced.

A somewhat similar formula, in which differences are employed instead of differential coefficients, is due to *Lubbock.* Expressing the derived functions in Woolhouse's formula in terms of the successive differences of $f_0, f_1, \dots f_r$ we obtain the formula in question,

$$\begin{aligned} f_0+f_{\frac{1}{m}}+f_{\frac{2}{m}}+\dots+f_r = m(f_0+f_1+\dots+f_r) &- \frac{m-1}{2}(f_0+f_r) \\ &- \frac{m^2-1}{12m}(\delta f_{r-\frac{1}{2}}-\delta f_{\frac{1}{2}}) - \frac{m^2-1}{24m}(\delta^2 f_{r-1}+\delta^2 f_1) \\ &- \frac{(m^2-1)(19m^2-1)}{720m^3}(\delta^3 f_{r-\frac{3}{2}}-\delta^3 f_{\frac{3}{2}}) \\ &- \frac{(m^2-1)(9m^2-1)}{480m^3}(\delta^4 f_{r-2}+\delta^4 f_2) \\ &+ \dots\dots \end{aligned}$$

Example 1.—To determine $\sum_{n=300}^{350}\frac{1}{n}$.

Let $m=10$, $w=10$.

Then from Woolhouse's formula we have

$$\begin{aligned} \sum_{n=300}^{350}\frac{1}{n} &= 10\left(\frac{1}{300}+\frac{1}{310}+\frac{1}{320}+\frac{1}{330}+\frac{1}{340}+\frac{1}{350}\right) \\ &\quad -\frac{9}{2}\left(\frac{1}{300}+\frac{1}{350}\right)-\frac{99}{12}\left(-\frac{1}{350^2}+\frac{1}{300^2}\right) \\ &= 0\cdot18512761-0\cdot02785714 \\ &\qquad\qquad\qquad\qquad 2432 \\ &= 0\cdot15724615 \end{aligned}$$

Using Lubbock's formula we get

$$\begin{aligned} \sum_{n=300}^{350}\frac{1}{n} &= 10(f_0+f_1+\dots+f_5)-\tfrac{9}{2}(f_0+f_5)-\tfrac{99}{120}(\delta f_{\frac{9}{2}}-\delta f_{\frac{1}{2}})-\tfrac{99}{240}(\delta^2 f_4+\delta^2 f_1) \\ &\quad -\tfrac{99\cdot1899}{720000}(\delta^3 f_{\frac{7}{2}}-\delta^3 f_{\frac{3}{2}})-\tfrac{99\cdot899}{480000}(\delta^4 f_3+\delta^4 f_2) \\ &= 0\cdot18512761-0\cdot02785714 \\ &\qquad\qquad\qquad\qquad 1938 \\ &\qquad\qquad\qquad\qquad 487 \\ &\qquad\qquad\qquad\qquad 3 \\ &\qquad\qquad\qquad\qquad 3 \\ &= 0\cdot15724616 \end{aligned}$$

To test the accuracy of these answers, the reciprocals of the numbers 300, 301, 302, ..., 350 were taken from Barlow's tables, and summed by means of a comptometer. The result was found to be 0·15724616. The closeness of the approximation is remarkable.

Example 2.—Obtain the value of $\sum_{n=100}^{175} \frac{1}{n}$.

Example 3.—By giving n increasing positive values, show that

$$\underset{n=\infty}{\mathrm{L}} \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ \dots\ +\frac{1}{n}-\log_e n\right)=0{\cdot}57721566\dots$$

30. Formulae of Approximate Quadrature.

The expansions which have been obtained in the preceding articles furnish the value of an integral to any required degree of accuracy, provided a sufficient number of terms is taken. We shall now consider certain formulae which are in their nature only *approximate*, and which, though frequently useful, cannot be used where great accuracy is required.

Suppose that it is required to integrate a function $f(x)$ between limits, which may, without loss of generality, be taken to be -1 and $+1$. We shall endeavour to obtain expressions of the type

$$H_0 f(h_0) + H_1 f(h_1) + \dots + H_n f(h_n)$$

which represent the integral as closely as possible, where $h_0, h_1, \dots h_n$ are $(n+1)$ values of x within the range of integration and $h_0, h_1, \dots h_n$, $H_0, H_1, \dots H_n$ do not depend on the function $f(x)$.

Let us first choose $H_0, H_1, \dots H_n$, so as to make the formula strictly accurate, so long as $f(x)$ is a polynomial of degree less than $n+1$.

Let $$F(x) = (x-h_0)(x-h_1)\dots(x-h_n).$$

Then $\dfrac{F(x)}{x-h_r}$ is a polynomial of degree less than $n+1$, so that the formula must be strictly accurate when for $f(x)$ we put $\dfrac{F(x)}{x-h_r}$. This gives

$$\begin{aligned}\int_{-1}^{+1} \frac{F(x)}{x-h_r}\,dx &= \sum_{r=0}^{n} H_r \frac{F(x)}{x-h_r} \\ &= H_r \underset{x \to h_r}{\mathrm{L}} \frac{F(x)}{x-h_r} \\ &= H_r F'(h_r).\end{aligned}$$

Hence, whatever h_0, h_1, ... h_n may be, in order that the above condition may be satisfied, the values of H_0, H_1, ... H_n must be given by

$$H_r = \frac{1}{F'(h_r)} \int_{-1}^{+1} \frac{F(x)}{x - h_r}\, dx.$$

In practice it is convenient to make the intervals between successive values of the argument equal to each other. Suppose, then, that h_0, h_1, ... h_n are chosen so as to divide the whole range of integration into n equal parts.

Then

$$h_0 = -1$$

$$h_1 = -1 + \frac{2}{n}$$

$$h_2 = -1 + \frac{4}{n}$$

$$\dots\dots\dots\dots\dots\dots$$

$$H_r = \frac{1}{(h_r - h_0)\dots(h_r - h_{r-1})(h_r - h_{r+1})\dots(h_r - h_n)}$$
$$\times \int_{-1}^{+1} (x - h_0)\dots(x - h_{r-1})(x - h_{r+1})\dots(x - h_n)\, dx$$

$$= \frac{1}{\frac{2r}{n} \cdot \frac{2(r-1)}{n} \dots \frac{2}{n} \cdot \frac{-2}{n} \cdot \frac{-4}{n} \dots \frac{-2(n-r)}{n}}$$
$$\times \int_{-1}^{+1} (x+1)\dots\left(x + 1 - \frac{2(r-1)}{n}\right)\left(x + 1 - \frac{2(r+1)}{n}\right)\dots(x-1)\, dx$$

$$= \frac{(-1)^{n-r}}{\left(\frac{2}{n}\right)^n r!\,(n-r)!}$$
$$\times \int_{-1}^{+1} (x+1)\dots\left(x + 1 - \frac{2(r-1)}{n}\right)\left(x + 1 - \frac{2(r+1)}{n}\right)\dots(x-1)\, dx$$

Now, let $t = \frac{n}{2}(x+1)$.

Then

$$H_r = \frac{(-1)^{n-r}\, 2}{n \,.\, r!\,(n-r)!} \int_0^n t(t-1)\dots(t-r+1)(t-r-1)\dots(t-n)\, dt.$$

Hence, finally, if we put

$$h_0 = a, \ \dots\ h_r = a + rw$$

we obtain

$$\int_a^{a+nw} f(x)\,dx = \sum_{r=0}^{n} H_r f(a+rw)$$

where

$$H_r = \frac{(-1)^{n-r} w}{r!\,(n-r)!} \int_0^n t(t-1)\ldots(t-r+1)(t-r-1)\ldots(t-n)\,dt.$$

This is known as *Cotes' Formula of Quadrature.* By means of it we may calculate the value of the integral without first of all forming a table of differences.

Special Cases.

(i.) Let $n=1$. Then

$$\int_a^{a+w} f(x)\,dx = H_0 f(a) + H_1 f(a+w)$$

where

$$H_0 = \frac{-w}{0!\;1!}\int_0^1 (t-1)\,dt = \tfrac{1}{2}w$$

$$H_1 = \frac{w}{1!\;0!}\int_0^1 t\,dt = \tfrac{1}{2}w.$$

Hence

$$\int_a^{a+w} f(x)\,dx = \frac{w}{2}\{f(a)+f(a+w)\}.$$

This is known as the *Trapezoidal Rule.*

(ii.) Let $n=2$. Then

$$\int_a^{a+2w} f(x)\,dx = H_0 f(a) + H_1 f(a+w) + H_2 f(a+2w)$$

where

$$H_0 = \frac{w}{0!\;2!}\int_0^2 (t-1)(t-2)\,dt = \tfrac{1}{3}w$$

$$H_1 = \frac{-w}{1!\;1!}\int_0^2 t(t-2)\,dt = \tfrac{4}{3}w$$

$$H_2 = \frac{w}{2!\;0!}\int_0^2 t(t-1)\,dt = \tfrac{1}{3}w.$$

Hence

$$\int_a^{a+2w} f(x)\,dx = \frac{w}{3}f(a) + \frac{4w}{3}f(a+w) + \frac{w}{3}f(a+2w).$$

This is called *Simpson's* * *Rule* or the *Parabolic Rule.*

* It was, however, first given by James Gregory.

Corollary.—If we divide the whole range of integration, say (a, b), into $2n$ equal parts, and apply Simpson's Rule to each of them, the value of the integral may be expressed as

$$I = \frac{b-a}{6n}\{y_0 + y_{2n} + 2(y_2 + y_4 + \dots + y_{2n-2}) + 4(y_1 + y_3 + \dots + y_{2n-1})\}$$

where $y_0, y_1, \dots y_{2n}$ are the values of $f(x)$ when x has the values $a, a + \frac{b-a}{2n}, \dots b$. This is Simpson's Rule in its generalised form. The rule may be stated thus :—

Divide the area into an even *number of strips by equidistant ordinates* $y_0, y_1, \dots y_{2n}$: *to the sum of the extreme ordinates add twice the sum of the other ordinates with* even *suffixes, and four times the sum of the ordinates with* odd *suffixes, and multiply this total by one-third of the common distance between the ordinates.**

(iii) Let $n = 3$. Then

$$\int_a^{a+3w} f(x)\,dx = H_0 f(a) + H_1 f(a+w) + H_2 f(a+2w) + H_3 f(a+3w)$$

where
$$H_0 = \frac{-w}{0!\,3!}\int_0^3 (t-1)(t-2)(t-3)\,dt = \tfrac{3}{8}w$$
$$H_1 = \frac{w}{1!\,2!}\int_0^3 t(t-2)(t-3)\,dt = \tfrac{9}{8}w$$
$$H_2 = \frac{-w}{2!\,1!}\int_0^3 t(t-1)(t-3)\,dt = \tfrac{9}{8}w$$
$$H_3 = \frac{w}{3!\,0!}\int_0^3 t(t-1)(t-2)\,dt = \tfrac{3}{8}w.$$

Hence

$$\int_a^{a+3w} f(x)\,dx = \tfrac{3}{8}w\{f(a) + 3f(a+w) + 3f(a+2w) + f(a+3w)\},$$

which is often termed the *Three-Eighths Rule.*

(iv) Let $n = 6$. Then it may be shown, as in the above special cases, that

$$\int_a^{a+6w} f(x)\,dx = \tfrac{1}{140}\{41f(a) + 216f(a+w) + 27f(a+2w) + 272f(a+3w) + 27f(a+4w) + 216f(a+5w) + 41f(a+6w)\}.$$

* The reader is warned that the notation varies in the different text-books, so that in all cases an examination of the notation used should first of all be made.

Adding

$$\tfrac{1}{140}\,\delta^6 f(a+3w) = \tfrac{1}{140}\{f(a) - 6f(a+w) + 15f(a+2w) - 20f(a+3w) + 15f(a+4w) - 6f(a+5w) + f(a+6w)\}$$

we obtain

$$\int_a^{a+6w} f(x)\,dx + \tfrac{1}{140}\,\delta^6 f(a+3w) = \tfrac{1}{140}\{42f(a) + 210f(a+w) + 42f(a+2w) + 252f(a+3w) + 42f(a+4w) + 210f(a+5w) + 42f(a+6w)\}$$
$$= \tfrac{3}{10}\{f(a) + 5f(a+w) + f(a+2w) + 6f(a+3w) + f(a+4w) + 5f(a+5w) + f(a+6w)\}.$$

Neglecting the term $\tfrac{1}{140}\,\delta^6 f(a+3w)$, which will be fairly small, we get *Weddle's Rule of Quadrature* for 6 intervals.

$$\int_a^{a+6w} f(x)\,dx = \tfrac{3}{10}\{f(a) + 5f(a+w) + f(a+2w) + 6f(a+3w) + f(a+4w) + 5f(a+5w) + f(a+6w)\}.$$

31. Comparison of the Accuracy of these Special Formulae.

We may compare the accuracy of these formulae by evaluating, by each of these methods, the definite integral $\int_1^7 \frac{dx}{x}$, the correct value of which is $\log_e 7 = 1{\cdot}94591$.

(i) Using the Trapezoidal Rule six times, *i.e.*, dividing the region into six equal portions and applying the rule to each of them, we get

$$\int_1^7 \frac{dx}{x} = \tfrac{1}{2}f(a) + f(a+w) + f(a+2w) + \dots + f(a+5w) + \tfrac{1}{2}f(a+6w)$$
$$= \tfrac{1}{2}\cdot 1 + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{2}\cdot\tfrac{1}{7}$$
$$= 2{\cdot}02143.$$

(ii) Using the Parabolic Rule repeated three times we get

$$\int_1^7 \frac{dx}{x} = \tfrac{1}{3}f(a) + \tfrac{4}{3}f(a+w) + \tfrac{2}{3}f(a+2w) + \tfrac{4}{3}f(a+3w) + \tfrac{2}{3}f(a+4w) + \tfrac{4}{3}f(a+5w) + \tfrac{1}{3}f(a+6w)$$
$$= \tfrac{1}{3}\left(1 + \tfrac{4}{2} + \tfrac{2}{3} + \tfrac{4}{4} + \tfrac{2}{5} + \tfrac{4}{6} + \tfrac{1}{7}\right)$$
$$= 1{\cdot}95873.$$

(iii) Applying the Three-Eighths Rule twice we have

$$\int_1^7 \frac{dx}{x} = \tfrac{3}{8}\{f(a)+3f(a+w)+3f(a+2w)+2f(a+3w)+3f(a+4w)$$
$$+3f(a+5w)+f(a+6w)\}$$
$$= \tfrac{3}{8}\{1+\tfrac{3}{2}+\tfrac{3}{3}+\tfrac{2}{4}+\tfrac{3}{5}+\tfrac{3}{6}+\tfrac{1}{7}\}$$
$$= 1{\cdot}96607.$$

(iv) Weddle's Rule gives

$$\int_1^7 \frac{dx}{x} = \tfrac{3}{10}\{1+\tfrac{5}{2}+\tfrac{1}{3}+\tfrac{6}{4}+\tfrac{1}{5}+\tfrac{5}{6}+\tfrac{1}{7}\}$$
$$= 1{\cdot}95286.$$

None of these gives the result with any high degree of accuracy, the errors being

(i.)	0·07552
(ii.)	0·01282
(iii.)	0·02016
(iv.)	0·00695

Example 1.—Evaluate $\int_1^7 \frac{dx}{x+5}$ by each of the methods of § 30.

Example 2.—Plot the curve $y^2=2x+25$ and find the area in square inches between the curve, the y-axis, and the double ordinate at $x=12$, the unit being one inch.

Example 3.—The ordinates of the boundary of the deck of a ship are 8, 30, 36, 40, 42, 42, 40, 38, 34, 28, and 8 feet respectively, and the common distance between them is 21 feet. Find the area of the deck.

Example 4.—The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour.

t	2	4	6	8	10	12	14	16	18	20
m/h	10	18	25	29	32	20	11	5	2	at rest.

Estimate the total distance run in 20 minutes.

32. Gauss' Method of Quadrature.

The accuracy of the method given in § 30 may be increased if we no longer suppose the intervals between the successive values $h_0, h_1, \ldots h_n$ of the argument to be equal.

Let us now endeavour to choose $H_0, H_1, \ldots H_n, h_0, h_1, \ldots h_n$ in such a way that the formula

$$\int_{-1}^{+1} f(x)\,dx = H_0 f(h_0) + H_1 f(h_1) + \ldots + H_n f(h_n)$$

shall be accurate for all functions $f(x)$ of degree equal to or less than $(2n+1)$.

Taking

$$f(x) = F(x)\,\phi(x)$$

where $\phi(x)$ denotes any polynomial of degree $<(n+1)$, and where

$$F(x) = (x-h_0)(x-h_1)(x-h_2)\dots(x-h_n)$$

we see that the condition to be satisfied may be written

$$\int_{-1}^{1} F(x)\,\phi(x)\,dx = 0.$$

Now it is well known that if $P_{n+1}(x)$ denotes the Legendre polynomial * of degree $(n+1)$, we have

$$\int_{-1}^{1} P_{n+1}(x)\,\phi(x)\,dx = 0$$

and, conversely, this condition is sufficient to determine the Legendre polynomials. Hence we have (save for a multiplicative constant)

$$F(x) = P_{n+1}(x)$$

and therefore the quantities $h_0, h_1, \dots h_n$ must be the roots of the equation

$$P_{n+1}(x) = 0.$$

The following are the values of the constants $h_0, h_1, \dots h_n, H_0, H_1, \dots H_n$ in the simplest cases.

(i) When $n=1$.

$$h_0 = -\frac{1}{\sqrt{3}} \qquad H_0 = 1$$

$$h_1 = \frac{1}{\sqrt{3}} \qquad H_1 = 1$$

(ii) When $n=2$.

$$h_0 = -\sqrt{\tfrac{3}{5}} \qquad H_0 = \tfrac{5}{9}$$

$$h_1 = 0 \qquad H_1 = \tfrac{8}{9}$$

$$h_2 = \sqrt{\tfrac{3}{5}} \qquad H_2 = \tfrac{5}{9}$$

---

* *Cf.* Whittaker and Watson's *Modern Analysis* or Byerly's *Fourier's Series and Spherical Harmonics.*

(iii) When $n=3$.

$$\begin{aligned} h_0 &= \sqrt{(\tfrac{3}{7}+\tfrac{2}{35}\sqrt{30})} & H_0 &= \tfrac{1}{2}-\tfrac{1}{36}\sqrt{30} \\ h_1 &= -\sqrt{(\tfrac{3}{7}-\tfrac{2}{35}\sqrt{30})} & H_1 &= \tfrac{1}{2}+\tfrac{1}{36}\sqrt{30} \\ h_2 &= \sqrt{(\tfrac{3}{7}-\tfrac{2}{35}\sqrt{30})} & H_2 &= \tfrac{1}{2}+\tfrac{1}{36}\sqrt{30} \\ h_3 &= \sqrt{(\tfrac{3}{7}+\tfrac{2}{35}\sqrt{30})} & H_3 &= \tfrac{1}{2}-\tfrac{1}{36}\sqrt{30} \end{aligned}$$

(iv) When $n=4$.

$$\begin{aligned} h_0 &= \sqrt{(\tfrac{5}{9}+\tfrac{2}{63}\sqrt{70})} & H_0 &= \frac{322-13\sqrt{70}}{900} \\ h_1 &= \sqrt{(\tfrac{5}{9}-\tfrac{2}{63}\sqrt{70})} & H_1 &= \frac{322+13\sqrt{70}}{900} \\ h_2 &= 0 & H_2 &= \frac{128}{225} \\ h_3 &= \sqrt{(\tfrac{5}{9}-\tfrac{2}{63}\sqrt{70})} & H_3 &= \frac{322+13\sqrt{70}}{900} \\ h_4 &= \sqrt{(\tfrac{5}{9}+\tfrac{2}{63}\sqrt{70})} & H_4 &= \frac{322-13\sqrt{70}}{900} \end{aligned}$$

Example 1.—Determine the value of $\int_{-1}^{+1}\frac{dt}{3+t}$.

(i) Using 3 ordinates

$$\int_{-1}^{+1}\frac{dt}{3+t} = \frac{5}{9}\cdot\frac{1}{3-\sqrt{\frac{3}{5}}}+\frac{8}{9}\cdot\frac{1}{3}+\frac{5}{9}\cdot\frac{1}{3+\sqrt{\frac{3}{5}}}$$
$$= 0{\cdot}69312165.$$

(ii) Using 5 ordinates

$$\int_{-1}^{+1}\frac{dt}{3+t} = \frac{322-13\sqrt{70}}{900}\cdot\frac{1}{3-\sqrt{(\frac{5}{9}+\frac{2}{63}\sqrt{70})}}+\frac{322+13\sqrt{70}}{900}\cdot\frac{1}{3-\sqrt{(\frac{5}{9}-\frac{2}{63}\sqrt{70})}}$$
$$+\frac{128}{225}\cdot\frac{1}{3}+\frac{322+13\sqrt{70}}{900}\cdot\frac{1}{3+\sqrt{(\frac{5}{9}-\frac{2}{63}\sqrt{70})}}$$
$$+\frac{322-13\sqrt{70}}{900}\cdot\frac{1}{3+\sqrt{(\frac{5}{9}+\frac{2}{63}\sqrt{70})}}$$
$$= 0{\cdot}693147157.$$

The correct value is 0·69314718, so that using 3 ordinates the error is in the 5th decimal place : using 5 ordinates, in the 8th decimal place.

Example 2.—Determine the values of (i) $\int_{-1}^{+1}\frac{dx}{x+2}$.

(ii) $\int_{-1}^{+1}\frac{dx}{1+x^2}$.

MISCELLANEOUS EXAMPLES.

1. The values of a function $f(x)$ for the values 1·050, 1·060, 1·070, 1·080, 1·090, 1·100 of the argument are 1·25386, 1·26996, 1·28619, 1·30254, 1·31903, 1·33565. Use Newton's formula of quadrature to show that the value of

$$\int_{1\cdot050}^{1\cdot100} f(x)\,dx$$

is 0·06472.

2. The ordinates of a plane curve are of the following lengths, 3·5, 4·7, 5·8, 6·8, 7·6, 8·1, 8·0, 7·2 feet, and they are 4 feet apart. What is the area included between the two end ordinates? and what is the distance of the centre of gravity of that area (i) from the extreme left-hand ordinate, and (ii) from the base line?

3. Approximate to the values of the following definite integrals :—

$$\text{(i)} \qquad \int_0^2 dx\sqrt{(3-x+x^3)}$$

$$\text{(ii)} \qquad \int_0^{\frac{\pi}{2}} dx\,\frac{\cos x}{\left(\frac{\pi}{2}-x\right)}$$

$$\text{(iii)} \qquad \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1-\frac{1}{10}\sin^2 x)}}.$$

4. Use (i) Lubbock's formula, (ii) Woolhouse's formula, to determine the value of an annuity-certain for 36 years, interest being at the rate of 3%.

5. Verify that the area of the curve $y=A+Bx+Cx^2+Dx^3$ between the limits $x=h$ and $x=-h$ is equal to the product of h into the sum of the ordinates at $x=h/\sqrt{3}$ and $x=-h/\sqrt{3}$.

In the case of the curve $y=f(x)=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5$ verify in like manner that the area between $x=h$ and $x=-h$ is equal to

$$\{5f(h\sqrt{\tfrac{3}{5}})+8f(0)+5f(-h\sqrt{\tfrac{3}{5}})\}h/9.$$

6. Corresponding values of x and y are given in the following table, the unit being the inch. Find the volume of the solid of revolution formed when this curve revolves about the x-axis. Find also the centre of gravity of this uniform solid.

x	0	4	8	12	16	20
y	0·50	2·58	4·41	5·40	5·10	0·50

7. A body of 200 lbs. is acted on by a variable force F. No other force acts on the body. When the body has passed through the distance x feet the force in pounds is as follows :—

x	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
F	20	21	21	20	19	18·5	18·0	13·5	9	4·5	0

Determine the work done upon the body from $x=0$ to $x=0·4$. If the velocity is 0 when $x=0$, what is its value when $x=0·4$?

8. The semi-ordinates in feet of the deck plan of a ship are respectively 3, 16·6, 25·5, 28·6, 29·8, 30, 29·8, 29·5, 28·5, 24·2, and 6·8, the common interval between them being 28 feet. Find the area of the deck.

9. The "half" ordinates in feet of the mid-ship section of a vessel are 12·5, 12·8, 12·9, 13, 13, 12·8, 12·4, 11·8, 10·4, 6·8, 0·5 respectively, and the common interval is 2 feet. Find the area of the whole section and the position of its centre of gravity.

10. The area of a ship's load-water plane is 8000 sq. ft. ; the body below the load-water plane is divided into six portions by equidistant horizontal sections, 3 feet apart, whose areas in sq. ft. are respectively 7600, 7000, 6000, 4500, 2800, and 250. Find (a) the displacement in tons, and (b) the number of tons which must be taken out of the ship to lighten her 4 inches.

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